MST204.4

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 4 Newtonian mechanics in one dimension

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Unit 4

Newtonian mechanics in one dimension

Prepared by the Course Team

The Open University

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Introduction

Think of an object that is moving in some complicated way. It might be a car accelerating on a motorway, a tennis ball flying through the air, a comet hurtling through space, or a pendulum swinging to and fro. Why does the object move as it does? How will it move in the future? To what extent can you influence its motion? Questions like these are clearly very important from a practical point of view. The control we exert over our environment depends, to a large extent, on our ability to find the right answers. Fortunately, mankind has discovered a systematic, organized body of knowledge called *mechanics* which can answer most questions about motion.

Mechanics can itself be divided into several distinct areas.

Quantum mechanics deals with the motion of very *small* objects (such as atoms, which have diameters of about 10^{-10} metres).

Relativistic mechanics deals with the motion of very *fast* objects (such as the electrons in a television tube, which have speeds of about 10^8 metres per second).

Newtonian mechanics is concerned with the more familiar everyday world of objects larger than atoms, which move at speeds less than a few million metres per second.

In this course we restrict our attention to *Newtonian mechanics*, but this still leaves a vast range of phenomena to discuss, including the motions of the car, tennis ball, comet and pendulum mentioned above. In fact, Newtonian mechanics is such a fertile area of mathematical modelling that we shall keep returning to it throughout the year, the present unit serving as an introduction to one of the major themes of the course.

Figure 1
Sir Isaac Newton (1642–1727)

If you are interested in learning more about Newton's life and work, a most readable survey is *Let Newton Be!*, edited by John Fauvel, Raymond Flood, Michael Shortland and Robin Wilson (Oxford University Press, 1988).

Isaac Newton was the great English mathematician whose name is given to this subject. His *Philosophiae Naturalis Principia Mathematica* of 1687 (Mathematical Principles of Natural Philosophy, or *Principia* for short) is one of the most celebrated examples of mathematical modelling. It was in *Principia* that Newton laid down the foundations of Newtonian mechanics. This great book, which showed for the first time how earthly and heavenly movements obey the same laws, is cast in the form of a set of propositions all deriving from three axioms, or *Laws of Motion*. It is these which still provide the basis for Newtonian mechanics; they are as follows.

Law I *Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.*

Law II *The change of motion is proportional to the motive force impressed, and is made in the direction of the straight line in which that force is impressed.*

Law III *To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

These laws did not spring fully-armed from Isaac Newton's imagination. Earlier investigators, notably Galileo and the French polymath René Descartes, had formulated some similar results. But it was Newton who perceived that these three were sufficient for the foundations of mechanics.

On a first reading, it may be hard to see what the laws are about, but do not let that discourage you! It will take much of this unit to explain the meaning of the first two and much of the course to explore their implications.

The third law is discussed in *Unit 17*.

Some points, however, are worth drawing to your attention immediately. It is perhaps surprising to notice that these seminal laws of mathematical modelling were written entirely in prose, with no hint of a mathematical symbol. The more symbolic form in which you will meet them later in the unit was developed during the eighteenth century. Newton himself followed through the complicated chains of reasoning arising from the laws with far less recourse to symbolism, or indeed to calculus, than later mathematicians have found necessary.

Notice also that whether one is dealing with these laws in prose, as Newton did, or converting them into mathematical symbols as we shall do, it is essential to know what the words *mean*. Newton's laws contain several terms which sound plausible enough

but whose translation into a form suitable for mathematical handling is less clear. Here are some of them:

uniform motion	straight line
force	change of motion
action	reaction.

You may like to read through the three laws again, highlighting or underlining those terms whose meaning you cannot immediately grasp, and then compare with this list.

Notice that we are not concerned yet with what the *laws* mean; that will come later. At present we are just noting that in order to model mathematically, the objects or events modelled have to be susceptible to mathematical handling. In short, words must be pinned down precisely, and the situation which they stand for may need to be somewhat idealized. One of the first aims of this unit is therefore to explain the mathematical interpretation of words such as *force* and *straight line*, and to explain the concept of *particle*, which is a very useful mathematical idealization of the notion of *body*. In the process you will see how these and the other concepts can be quantified. After that you can begin to attain a fuller understanding of Newton's laws themselves and discover why they are so valuable.

Study guide

The five sections in this unit should be studied in the order that they appear. The first section is concerned with concepts like position, velocity and acceleration, which describe the way an object moves. The second section (with television) discusses the meaning of force and introduces Newton's laws of motion which, in any given situation, predict the motion that actually occurs. The third and fourth sections (both with audio-tape) give you practice in applying Newton's laws to solve problems, in the context of the motion of falling objects. Finally, Section 5 contains a further selection of exercises which can be used for additional practice before the TMA or for later revision.

You should bear in mind that this unit is only an introduction to the ideas of mechanics. Later units will concentrate on using these ideas to model real situations, but before becoming immersed in details it is important to appreciate the concepts and laws that are inherent in Newton's way of understanding the world. The framework of mechanics presented in this unit is of great significance. The claims made by Newton's laws are far more ambitious than those made by, say, the logistic equation for fish populations (*Unit 3*, Section 2). The factors affecting fish populations are so diverse and ill-understood that no one would be very surprised if the herring population in a particular ocean failed to conform to the logistic equation. But a failure of Newton's laws would affect our whole conception of the physical world.

The television programme is a crucial part of the week's work, as it introduces Newton's laws of motion. The programme notes in Subsection 2.2 are quite full, so that you should be able to make some progress if you reach this point in the unit with time to spare before the scheduled broadcast. Nevertheless, you should still ensure that you view the programme, because the ideas presented there are central to the unit.

1 The description of motion

1.1 The concept of a particle

The motion of a real object, say a leaf that is falling to the ground, is very difficult to describe exactly. The leaf may rotate, bend or vibrate while moving along a complicated path in three-dimensional space. It would be foolhardy to meet all of these difficulties head-on. *This unit therefore concentrates on the simple case of a particle moving along a straight line.* Later in the course you will see how the ideas presented here can be extended to more complicated situations.

It is important to understand the meaning that is attached to the word 'particle' in mechanics. We have all seen tiny scraps of matter such as pollen and grains of sand. A particle is an idealization of such objects. We define a **particle** to be a material object whose size and internal structure are negligible.

Neglecting an object's size means that its location at any given time may be completely described by a single point in space, and that it moves along a single curve. Neglecting the internal structure of an object, and hence also any internal motion, amounts to saying that the curve described in time by the particle model gives the only information of interest about the way in which the object moves.

We tend to think of particles as modelling small physical objects but, in deciding whether the particle model is appropriate, smallness must be assessed relative to the physical context under consideration. For example, the Earth's diameter is about 1.3×10^7 metres, while the distance between the Earth and the Sun is about 1.5×10^{11} metres. We normally think of the diameter of the Earth as being large, but it is small by comparison with the distance to the Sun, and in order to predict the Earth's orbit around the Sun it is quite satisfactory to model the Earth by a particle. On the other hand this model is inadequate for the prediction of eclipses between the Earth, Moon and Sun, as here the diameters of the Earth and Moon are crucial factors.

Whether a particle-based model will be satisfactory is not just a question of size. For example, if a ball is placed on a rough sloping table then it will roll down the slope. A particle model could be used to describe the trajectory of the ball's centre, but it would not be adequate to keep track of the rolling motion that takes place about the centre. This inadequacy of the particle model occurs regardless of the ball's size, since the same consideration would apply to a football, a tennis ball, a marble or a ball bearing. The rolling arises initially because the centre of the ball is not vertically above the point of contact with the table, and the distance between these two points (which is the radius of the ball) is important in determining how much rolling takes place.

You might think from this last example that the simple particle model is of very limited use, but in fact the example hints at how that simple model can be extended. You will see in a later unit that the motion of an object can be described well by specifying

(a) the motion of a particular point within the object, called its *centre of mass*;

(b) the motion of the whole object relative to its centre of mass.

The motion of the centre of mass may be predicted by considering a particle of the same mass as the object placed at that point and subjected to all of the external forces which act upon the object. So even in this more refined model the concept of a particle is important. Alternatively, it may be appropriate to think of an object as being composed of a number of elements, each of which can be modelled individually by a particle.

This result will be derived in *Unit 17*.

Having pointed out the importance of particles in later developments, we shall in this unit concentrate on the particle model alone. This is in accordance with the general modelling principle of starting with the simplest available model. To simplify matters further we consider here only one-dimensional motion, in which the particle moves along a single straight line. As suggested above, the position of the particle on this line can be thought of as the centre of mass of the object being modelled, which for a symmetric object such as a ball will be the geometric centre.

1.2 The position of a particle

In this unit, as well as in *Units 7* and *8*, we concentrate on the motion of a particle in one dimension. This may seem somewhat restrictive, but you will see in the latter half of the course how the ideas and techniques introduced here can be extended to three-dimensional motion. Typical of the questions to be considered in this unit are the following:

> If a marble is dropped from the Clifton Suspension Bridge, how long does it take to fall into the River Avon below? And what is its velocity just before it hits the water?

To answer these questions we need to find expressions for the position and velocity of the falling marble. In the rest of this section you will see how the concepts of *position*, *velocity* and *acceleration* are defined, and what connections between them are provided by the calculus. After that we turn to the question of what causes and influences the motion of an object.

The position of a particle on a line at a given instant of time is a single point (see Figure 1). In order to define the position of a physical object, which has a finite size, we can use the position of a representative point of the object, such as the geometric centre of our marble. In order to quantify the concept of position, we choose an x-axis along the line as shown in Figure 2, with an origin O and a scale (measured in metres, say). The position of a particle on this x-axis can then be specified by giving the number, x, which labels the point occupied by the particle. Note that a positive value of x represents a position which is in the direction of increasing x from the origin, whereas a negative value of x represents a position on the other side of the origin. In Figure 2 the x-axis is horizontal and points to the right. However, it could equally well have been chosen to point in any other direction. This choice will usually depend on what motion is being modelled. For the falling marble, for example, a vertical x-axis would be appropriate.

Figure 1

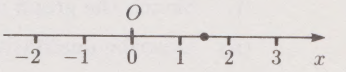


Figure 2

Exercise 1

What are the positions x of particles A, B and C in Figure 3 below, and what are the distances between them?

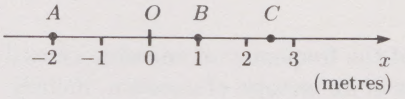


Figure 3

[*Solution on page 49*]

As time passes, the particle may change its position on the line. In order to quantify *time*, we need a clock which is calibrated in, say, seconds and a definite origin of time at which the clock is set to read zero. This origin can be chosen as any convenient instant, such as the moment when the marble is dropped from the Clifton Suspension Bridge. Once we have chosen our origin and unit of calibration, any instant of time can be specified by the clock reading, t. Again note that t will be positive for instants after our chosen origin of time, whereas t will be negative for instants before the origin.

If a particle moves along a line, then at any time t the particle has a well-defined position x which depends on the value of t. In other words, the position of the particle is a function of t. This function, denoted by $x(t)$, is known as the **position function** of the particle.

$x(t)$ means the value of the variable x at time t.

Example 1

The motion of a particle is described by the position function

$$x = 4 - (t-1)^2 \qquad (0 \le t \le 4),$$

where x is measured in metres and t in seconds.

(i) Sketch the graph of the position function.

(ii) What is the position of the particle at times $t = 0$, $t = 1$, $t = 2$, $t = 3$ and $t = 4$?

(iii) Describe qualitatively how the position of the particle varies with time.

Solution

(i) See Figure 4.

(ii) The particle's positions at the given times are $x(0) = 3$, $x(1) = 4$, $x(2) = 3$, $x(3) = 0$ and $x(4) = -5$.

(iii) At $t = 0$ the particle is at the position $x = 3$. Subsequently it moves in the direction of increasing x (that is, away from the origin) until it reaches $x = 4$ at time $t = 1$. After this instant, it moves in the direction of decreasing x, passing through the origin $x = 0$ at $t = 3$, and finally reaching $x = -5$ at $t = 4$. □

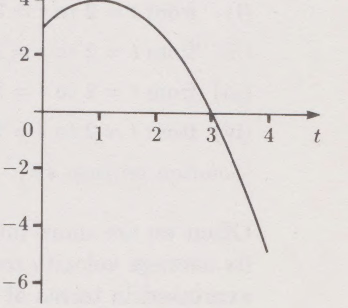


Figure 4

Exercise 2

Consider the motion of a particle whose position function is

$$x = \begin{cases} t & 0 \le t < 1, \\ 1 & 1 \le t < 2, \\ 1 - (t-2)^2 & 2 \le t \le 3, \end{cases}$$

where x is measured in metres and t in seconds.

(i) Sketch the graph of the position function.

(ii) Describe qualitatively how the position of the particle varies with time.

[*Solution on page 49*]

In this course we shall usually measure quantities in the SI system (the Système International d'Unités), which is widely accepted as a standard throughout science and technology. The details of the SI system are explained in the Handbook.

Prior to 1967 the SI unit of length, the *metre*, was based on a bar of a platinum-iridium alloy stored at the Bureau of Metric Standards, Sèvres, France. Since 1983 the metre has been specified in terms of the velocity of light, being the distance that light travels in exactly 1/299 792 458 of a second.

The SI unit for length is the metre, abbreviated as m.

The SI unit of time, the *second*, is defined in terms of the frequency of emission caused by the transition between two hyperfine energy levels of an isotope of caesium, namely ^{133}Ce. (This is the so-called 'atomic clock'.) Until 1967 the second was defined in terms of the Earth's rotational period. However, modern technology has demonstrated that, for various reasons, some days are shorter than others, and that on average the period of the Earth is gradually becoming longer as the centuries pass.

The SI unit for time is the second, abbreviated as s.

1.3 The velocity of a particle

Once the position function of a particle has been specified, everything, in principle, is known about the particle's motion. However, other quantities, such as the speed and acceleration, are often of more immediate interest. For example, an aggressive motorist might be proud of his acceleration away from traffic lights, whereas a policeman would probably be more interested in the motorist's speed.

The *average velocity* of a car over a given time period is relatively simple to calculate, being defined by

$$\text{average velocity} = \frac{\text{distance travelled}}{\text{time taken}}.$$

In the time interval from $t = t_0$ to $t = t_0 + \tau$, the distance travelled is $x(t_0 + \tau) - x(t_0)$, and so the average velocity over this time interval is

$$\frac{x(t_0 + \tau) - x(t_0)}{(t_0 + \tau) - t_0} = \frac{x(t_0 + \tau) - x(t_0)}{\tau}.$$

Exercise 3

The position x of a particle at time t is given by the position function

$$x(t) = t^2 + t.$$

Calculate the average velocity of this particle during each of the following time intervals:

(i) from $t = 2$ to $t = 3$;

(ii) from $t = 2$ to $t = 2.1$;

(iii) from $t = 2$ to $t = 2.01$;

(iv) from $t = 2$ to $t = 2.001$.

[*Solution on page 49*]

It will be assumed from here on that, unless stated otherwise, distance and position are specified in metres and time in seconds.

Often we are more interested in the velocity of a particle at a given instant rather than its average velocity over an extended time interval. For example, speed limits are expressed in terms of instantaneous velocities, and this is what car speedometers indicate. Exercise 3 suggests that the velocity v at a given instant can be estimated better and better by evaluating the average velocity over smaller and smaller intervals

of time. Taking this process to its limit, we define the instantaneous **velocity** of a particle at time t to be

$$v(t) = \lim_{\tau \to 0} \left[\frac{x(t+\tau) - x(t)}{\tau} \right].$$

In SI units velocity is measured in metres per second, abbreviated as $\mathrm{m\,s^{-1}}$.

You should recognize the right-hand side of this equation as the definition of the derivative of the position function, namely $x'(t)$. In Leibniz notation this derivative is denoted by dx/dt. In mechanics we often use a third notation, with a dot, to denote a differentiation which is specifically with respect to time. Thus here we would write $v = \dot{x}$. So the (instantaneous) velocity of a particle at time t is defined as the derivative of the position function,

$$v(t) = x'(t) = \frac{dx}{dt} = \dot{x}(t).$$

As t varies, the velocity v is specified as a function of t, and this function $v(t)$ is called the **velocity function** of the particle.

Example 2

The position function of a particle is

$$x(t) = t^2 + t.$$

Find the velocity function of the particle, and calculate the velocity at the instant $t = 2$.

Solution

By definition

$$v(t) = \frac{dx}{dt} = 2t + 1.$$

So $v(2) = 2 \times 2 + 1 = 5$, and the velocity of the particle at $t = 2$ is $5\,\mathrm{m\,s^{-1}}$.

(Notice that this agrees with the answers to Exercise 3, which seemed to be tending to the limiting value 5 for small time intervals.) □

Exercise 4

A particle is set in motion at time $t = 0$, and its position thereafter is given by the position function

$$x(t) = t^2 - 4t + 1 \qquad (t \geq 0).$$

(i) Calculate the velocity of the particle after 4 seconds.

(ii) Calculate the velocity of the particle after 1 second.

(iii) Find those times at which the particle is instantaneously at rest.

[*Solution on page 49*]

The definition of the velocity as a derivative can be interpreted geometrically as the slope of the tangent to the graph of the position function, as shown in Figure 5.

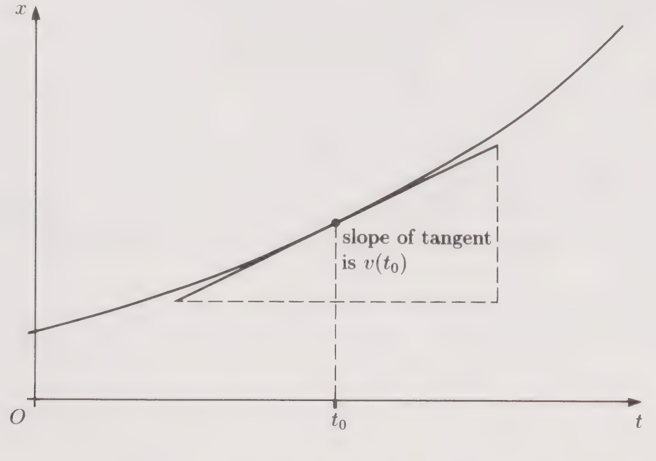


Figure 5

Notice that if the particle is moving in the direction of increasing x then the position function will be an increasing function and the velocity will be positive. On the other hand, if the particle is moving in the opposite direction then the position function will be a decreasing function and the velocity will be negative (see Figures 6 and 7). For example, in Exercise 4 you found that the velocity was positive at $t = 4$, and so at that time the particle was moving in the direction of increasing x. At the instant $t = 1$ the velocity was negative, and so at that moment the particle was moving in the direction of decreasing x.

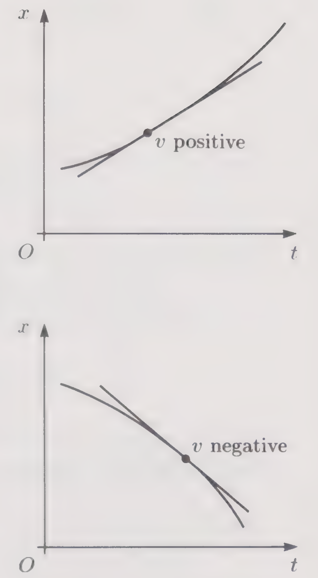


Figure 6

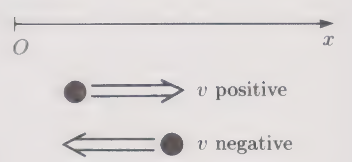


Figure 7

For one-dimensional motion, therefore, the sign of the velocity indicates the direction of the particle's motion. Sometimes we are interested only in the *magnitude* of the velocity, which is called the **speed** of the particle. So we have

$$\text{speed} = |v(t)| = \begin{cases} v(t) & \text{if } v(t) \geq 0, \\ -v(t) & \text{if } v(t) < 0. \end{cases}$$

The mathematical definition of velocity as the rate of change of position does not always accord with the colloquial use of the word, where it is sometimes taken to be synonymous with speed. Therefore, when translating a statement such as 'the car was travelling with velocity 50 kilometres per hour' into mathematical language, it is important to assign to v the correct sign for the corresponding direction of motion.

Exercise 5

The position function of a particle is

$$x(t) = t^3 - 12t.$$

(i) Find the velocity and speed of the particle at each of the instants $t = 1$, $t = 2$ and $t = 3$.

(ii) Sketch the graphs of the velocity function and the speed function.

[*Solution on page 49*]

1.4 The acceleration of a particle

In many situations velocity is less important than *changes* in velocity. For example, if you are on board a train, travelling at a steady speed, you may not even notice that you are moving. You will have no difficulty in, say, drinking a cup of tea. However, this operation becomes more hazardous if the driver changes the velocity of the train by putting on the brakes! In this case, the rate of change of velocity is an important factor.

You have just seen that the velocity $v(t)$ of a particle is defined as the rate of change of the particle's position; it is calculated by differentiating the position function $x(t)$. In an exactly similar way, the **acceleration** $a(t)$ of a particle is defined as the rate of change of the particle's velocity and is calculated by differentiating the velocity function $v(t)$. That is,

$$a(t) = \lim_{\tau \to 0} \left[\frac{v(t+\tau) - v(t)}{\tau} \right] = \frac{dv}{dt} = v'(t) = \dot{v}(t).$$

The SI units for acceleration are metres per second per second, abbreviated as $\mathrm{m\,s^{-2}}$.

In other words, the acceleration of a particle is just the rate at which its velocity increases with time. Since velocity is defined as dx/dt, we can also write

$$a(t) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = x''(t) = \ddot{x}(t),$$

where the double dot notation represents a second derivative with respect to time. So the acceleration can be found either by differentiating the velocity function *once*, or by differentiating the position function *twice*. This means that you can think of acceleration either as the *slope* of a graph of v against t (see Figure 8), or as the *rate of increase of slope* of a graph of x against t.

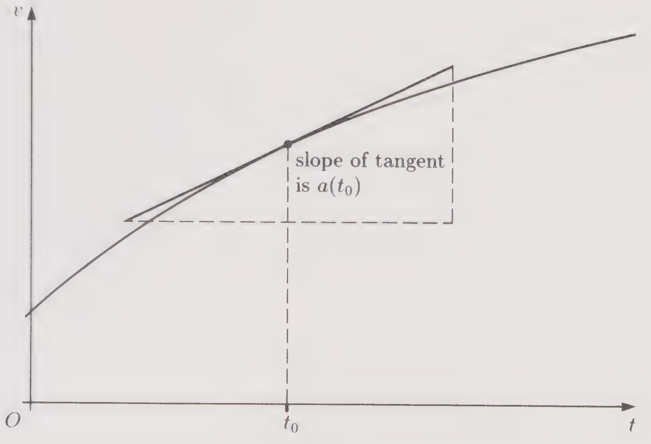


Figure 8

Example 3

The position of a particle at time t is specified by the position function

$$x(t) = 7 + 3t - 10t^2 + 3t^3.$$

(i) Calculate the velocity and speed of the particle at $t = 2$.

(ii) Calculate the acceleration of the particle at $t = 2$.

Solution

(i) The first derivative of the position function is the velocity:

$$v(t) = \frac{d}{dt}(7 + 3t - 10t^2 + 3t^3)$$
$$= 3 - 20t + 9t^2.$$

At $t = 2$, we have

$$v(2) = 3 - 40 + 36 = -1.$$

Therefore the velocity at $t = 2$ is $v(2) = -1\,\mathrm{m\,s^{-1}}$, and the speed is

$$|v(2)| = 1\,\mathrm{m\,s^{-1}}.$$

(ii) The acceleration is found by differentiating the velocity function:

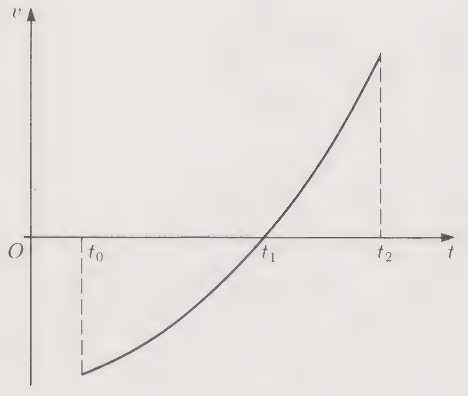
$$a(t) = \frac{d}{dt}(3 - 20t + 9t^2)$$
$$= -20 + 18t.$$

At $t = 2$ this gives

$$a(2) = -20 + 36 = 16,$$

so the acceleration at $t = 2$ is $16\,\mathrm{m\,s^{-2}}$. □

Notice that our definition of acceleration as the rate of increase of *velocity* differs from its everyday meaning as the rate of increase of *speed*. (In everyday language also, deceleration is the rate of decrease of speed.) According to its mathematical definition, a change of velocity will produce an acceleration which may be positive or negative. A positive acceleration arises if the velocity is increasing, but this may correspond either to an increase or to a decrease in speed, depending on whether the particle is moving in the direction of increasing x or of decreasing x. For example, Figure 9 shows on the left the graph of a velocity function for which the acceleration is positive during the whole time interval $t_0 < t < t_2$. However, the graph on the right demonstrates that the corresponding speed of the particle is decreasing for the interval $t_0 < t < t_1$!



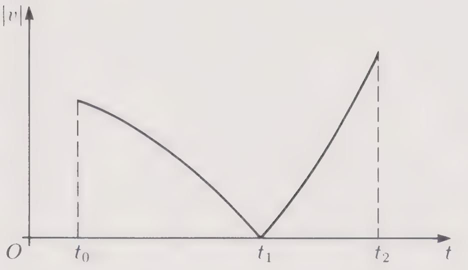


Figure 9

Similar considerations apply if the acceleration is negative. In this case the particle is either moving in the direction of increasing x (v positive) but slowing down, or moving in the opposite direction (v negative) and speeding up.

Exercise 6

The position function of a particle is given by

$$x(t) = te^{-t} \qquad (t > 0).$$

(i) Find the velocity and acceleration functions.

(ii) Sketch the graphs of the position, velocity and acceleration functions.

(iii) Identify a time interval during which the acceleration is positive though the speed is decreasing.

[*Solution on page 49*]

1.5 Relationships between position, velocity and acceleration

We start this subsection by recalling the major definitions for particle motion which were introduced earlier.

The definitions of velocity and acceleration for one-dimensional motion

The motion of a particle along a straight line is described by a *position function* $x(t)$. For each instant t in time, this function specifies the position $x(t)$ which is occupied by the particle.

The *velocity* of the particle at time t is the rate of increase of its position,

$$v(t) = \frac{dx}{dt}.$$

The *acceleration* of the particle at time t is the rate of increase of its velocity,

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

If the position function of a particle is known then its velocity and acceleration can be calculated directly by differentiation, using the relations $v(t) = dx/dt$ and $a(t) = dv/dt$. However, in mechanics we often have the opposite situation, in which the acceleration of a particle is a known function of time and we wish to calculate the velocity and position functions. This can be achieved by integration, since

Strictly speaking, constants of integration are included implicitly within indefinite integrals. However, we have chosen to write them as separate terms here for clarity.

$$v = \int \frac{dv}{dt}\,dt + c_1 = \int a(t)\,dt + c_1$$

and

$$x = \int \frac{dx}{dt}\,dt + c_2 = \int v(t)\,dt + c_2.$$

Here c_1 and c_2 are constants of integration which can often be found using given initial conditions. The method is best illustrated by considering an example and some exercises.

Example 4

The acceleration a of a particle at time t is given by

$$a(t) = 12t^2 + 2.$$

At the instant $t = 1$ the particle is at $x = 3$ and has velocity $v = 2$.

(i) Find the velocity function $v(t)$ and the position function $x(t)$ of the particle.

(ii) Find the position and velocity of the particle at time $t = 2$.

Solution

(i) Since the acceleration function is known, we can write down a first-order differential equation for the velocity,

$$a = \frac{dv}{dt} = 12t^2 + 2.$$

This equation can be solved by direct integration, giving

$$\begin{aligned} v &= \int (12t^2 + 2)\,dt + c_1 \\ &= 4t^3 + 2t + c_1. \end{aligned}$$

To find the value of the constant of integration c_1, we use the fact that the particle has velocity $v = 2$ at the instant $t = 1$, so that

$$2 = 4 + 2 + c_1, \quad \text{or} \quad c_1 = -4.$$

Hence the velocity function is

$$v(t) = 4t^3 + 2t - 4.$$

This result allows us to write down a first-order differential equation for the position function, namely

$$\frac{dx}{dt} = 4t^3 + 2t - 4.$$

Solving this equation by direct integration gives

$$\begin{aligned} x &= \int (4t^3 + 2t - 4)\,dt + c_2 \\ &= t^4 + t^2 - 4t + c_2. \end{aligned}$$

The initial condition $x = 3$ at $t = 1$ can be used to find the constant of integration c_2. This leads to

$$3 = 1 + 1 - 4 + c_2, \quad \text{or} \quad c_2 = 5.$$

Hence the position function is

$$x(t) = t^4 + t^2 - 4t + 5.$$

(ii) At $t = 2$ the position and velocity functions have values

$$x(2) = 2^4 + 2^2 - 4 \times 2 + 5 = 17$$

and $$v(2) = 4 \times 2^3 + 2 \times 2 - 4 = 32.$$

Hence at time $t = 2$ the particle has position 17 m and velocity 32 m s^{-1}. □

Exercise 7

A particle moves along a straight line upon which an x-axis has been defined. At time t the particle has an acceleration given by

$$a(t) = 18t - 20 \qquad (t \geq 0).$$

Initially, at $t = 0$, the particle has position $x(0) = 7$ and velocity $v(0) = 3$. Find the position of the particle at time $t = 10$.

Exercise 8

The acceleration function of a particle moving along a straight line with defined x-axis is

$$a(t) = e^{-kt} \qquad (t \geq 0),$$

where k is a constant. Initially, at $t = 0$, the particle is at the origin ($x(0) = 0$) and is at rest ($v(0) = 0$). Find the position and velocity functions of the particle.

[*Solutions on page 50*]

In the remainder of this section we ask you to consider two simple situations of particular interest. The first of these is *uniform motion*, that is, motion in which the velocity always has a constant value.

Exercise 9

Consider a particle moving along a straight line with constant velocity v_0, that is, with velocity function

$$v(t) = v_0.$$

(i) Find the acceleration function of the particle.

(ii) If initially, at $t = 0$, the particle is at the position $x(0) = x_0$, find the position function.

(iii) Sketch the graphs of the position, velocity and acceleration functions.

[*Solution on page 50*]

The results of Exercise 9 are summarized below.

Uniform motion

When a particle is in uniform motion the velocity is constant and the acceleration is zero. The position, velocity and acceleration functions are given by

$$x = x_0 + v_0 t,$$
$$v = v_0,$$
$$a = 0,$$

where x_0 is the position at time $t = 0$ and v_0 is the value of the constant velocity. Typical graphs of the functions $x(t)$, $v(t)$ and $a(t)$ are shown in Figure 10.

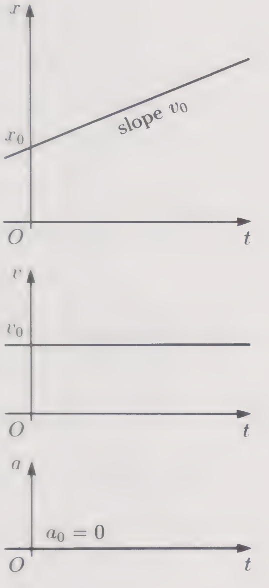


Figure 10

In Exercises 10 and 11 we ask you to consider motion in which the acceleration is constant.

Exercise 10

A particle moves along a straight line with constant acceleration a_0.

(i) Show that the velocity v and position x of the particle at time t are given by

$$v(t) = v_0 + a_0 t,$$
$$x(t) = x_0 + v_0 t + \tfrac{1}{2} a_0 t^2,$$

where $v(0) = v_0$ and $x(0) = x_0$.

(ii) By eliminating t between these two equations, show that

$$v^2 = v_0^2 + 2a_0(x - x_0).$$

[*Solution on page 50*]

The equation relating v and x which you obtained in Exercise 10(ii) has considerable physical significance, as you will see in *Unit 7*. This equation may be derived more directly by employing an alternative expression for the acceleration. In fact, using the chain rule, the acceleration can be written as

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}.$$

You will see later in the unit that this formula is particularly useful when the acceleration is a known function of position or velocity.

Exercise 11

A particle moves along a straight line with constant acceleration a_0. Initially it is at position x_0 and has velocity v_0.

Use the expression

$$a = v\frac{dv}{dx}$$

for the acceleration to derive the equation

$$v^2 = v_0^2 + 2a_0(x - x_0).$$

[*Solution on page 50*]

The relationships which you obtained in Exercises 10 and 11 are summarized below for completeness. However, should you require these equations for a particular problem then it is better to derive them from first principles rather than to try and memorize them.

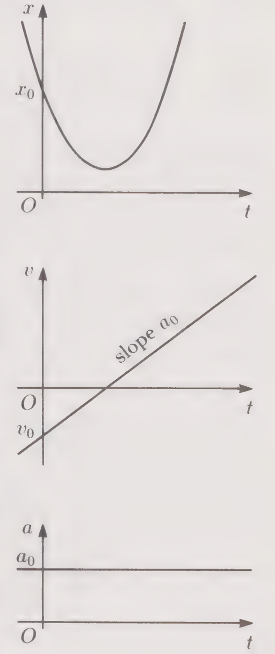


Figure 11

Motion with constant acceleration

The position x and velocity v for a particle motion with constant acceleration a_0 satisfy

$$x = x_0 + v_0 t + \tfrac{1}{2}a_0 t^2,$$
$$v = v_0 + a_0 t,$$
$$v^2 = v_0^2 + 2a_0(x - x_0),$$

where x_0 and v_0 are the position and velocity at time $t = 0$.

Warning: the above formulas are applicable only when the acceleration is constant.

Typical graphs of the functions $x(t)$, $v(t)$ and $a(t)$ are shown in Figure 11.

Summary of Section 1

1. This unit is concerned with objects that can be modelled by particles moving in straight lines. A **particle** is a material object whose size and internal structure are negligible, so that at any given time it is located at a single point.

2. The motion of a particle along a straight line is described by a position function $x(t)$. For each instant t in time, this function specifies the position $x(t)$ occupied by the particle.

3. The **velocity** v of the particle is the derivative of the position function with respect to time, that is,

$$v = \frac{dx}{dt}.$$

4. The **acceleration** a of the particle is the derivative of the velocity function with respect to time. In symbols,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}.$$

5. In the special case of **uniform motion**, by which is meant motion with constant velocity v_0, we have

$$x = x_0 + v_0 t,$$
$$v = v_0,$$
$$a = 0,$$

where x_0 is the position at time $t = 0$.

6. In the special case of motion with **constant acceleration** a_0, we have

$$x = x_0 + v_0 t + \tfrac{1}{2} a_0 t^2,$$
$$v = v_0 + a_0 t,$$
$$v^2 = v_0^2 + 2a_0(x - x_0),$$

where x_0 and v_0 are the position and velocity of the particle at the instant $t = 0$.

2 Newton's laws of motion

2.1 Towards a theory of motion

Section 1 introduced the basic concepts of position, velocity and acceleration, which are needed to *describe* the motion of a particle. In this section we shall introduce two more concepts, *force* and *mass*, which allow us to go beyond the mere description of motion and formulate laws *predicting* what motions take place.

At first sight you might expect that different sets of rules would be required for each type of object, one set for tennis balls, another set for planets, and so on. Fortunately there is a simple underlying pattern. Newton was able to see beyond individual cases, and his three laws of motion form a framework or theory for predicting the motion of all material objects, from pollen grains to planets. The purpose of this section is to explain the meaning of Newton's first and second laws in the context of a particle moving along a straight line. In the television programme you will see experiments that demonstrate the validity of Newton's laws. The programme notes are in Subsection 2.2. There is a small amount of text to read and three exercises to be attempted before viewing the programme.

Newton's third law will be discussed later in the course.

In order to explain what is meant by a theory of motion, we shall focus on a specific example. Consider a toboggan on a horizontal icy surface such as a frozen lake. Left undisturbed, the toboggan remains static; it must be pushed or pulled in some way if it is to be set in motion. We say that a *force* is needed to start the toboggan moving, and in the absence of such a force the toboggan can remain permanently at rest. However, if you give the toboggan a push and then release it, the toboggan will move across the ice at almost constant velocity in the direction that it has been pushed. This suggests that under ideal conditions the following applies:

> in the absence of a force, the toboggan remains at rest or moves with constant speed in a straight line.

Galileo (1564–1642) was the first person to propose that the natural state of an object in the absence of a force is uniform motion, rather than rest.

The toboggan does eventually slow down, but this is due to air resistance and friction between the toboggan runners and the ice. In competitive tobogganing, the tobogganers go to great lengths to reduce these resistive forces by streamlining the toboggan and waxing the runners.

Now suppose that you apply a force by pushing the toboggan. You cannot quantify this force, but the sensations in your muscles and nerves will reveal whether you are pushing gently or firmly. From experience you know that:

> the harder you push, the further and faster the toboggan moves in a given time.

This suggests that there is a link between the force that is applied and the way in which the toboggan moves.

Next, imagine pushing two identical toboggans, one of which is empty while the other carries a heavy person. If you apply the same force to the two toboggans then the laden toboggan will move more sluggishly. To achieve the same motion in each case it is necessary to apply a greater force to the laden toboggan. In other words:

> if you apply the same force to the two toboggans, the laden toboggan does not travel as far or as fast in a given time as the empty toboggan;
>
> in order for the two toboggans to move in the same way, a greater force must be applied to the laden toboggan than to the empty toboggan.

In general, it seems that three concepts are linked together:

1. the *force* that is applied to an object;
2. the amount of *matter* which the object contains;
3. the *motion* of the object.

Newton proposed in *Principia* that this link takes the form

$$\text{force} = \text{mass} \times \text{acceleration}.$$

The television programme explains the meaning of this equation, demonstrates that it holds in practice and uses it to analyse the results of some simple experiments.

Exercise 1

(i) A car on a flat road requires a motive force in order to maintain a constant speed; if the engine is switched off then the car slows down. It might be thought from the above that if an object is moving with constant velocity then there is no force acting on it. Try to explain this apparent contradiction.

(ii) A toboggan on an icy slope may accelerate, even when it is not being pushed. From the above it may be implied that a force is necessary to cause acceleration. Try to identify the force involved in this case.

[*Solution on page 50*]

2.2 Newton's second law of motion (Television Subsection)

Attempt the following two exercises before viewing the television programme.

Exercise 2

A particle has constant acceleration g, and initially, at time $t = 0$, it is at rest ($v = 0$) at the origin ($x = 0$). Show that the velocity v and position x of the particle at time t are given by

$$v = gt, \qquad x = \tfrac{1}{2}gt^2.$$

Exercise 3

Verify by differentiation and substitution that the solution of the differential equation

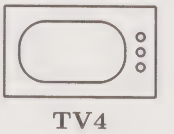
$$m\frac{dv}{dt} = mg - kv$$

(where m, g and k are constants) which satisfies the initial condition $v = 0$ when $t = 0$ is

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$

[*Solutions on page 51*]

Now watch the television programme.



TV4

Read the following notes after viewing the programme.

The first part of the programme consisted of five experiments and their interpretation. These experiments are not the most accurate that science can devise, but they do illustrate clearly the central equation in Newtonian mechanics:

$$\text{force} = \text{mass} \times \text{acceleration}$$

or

$$F = ma.$$

This equation is referred to as **Newton's second law**, and the special case of zero force and zero acceleration is known as **Newton's first law**.

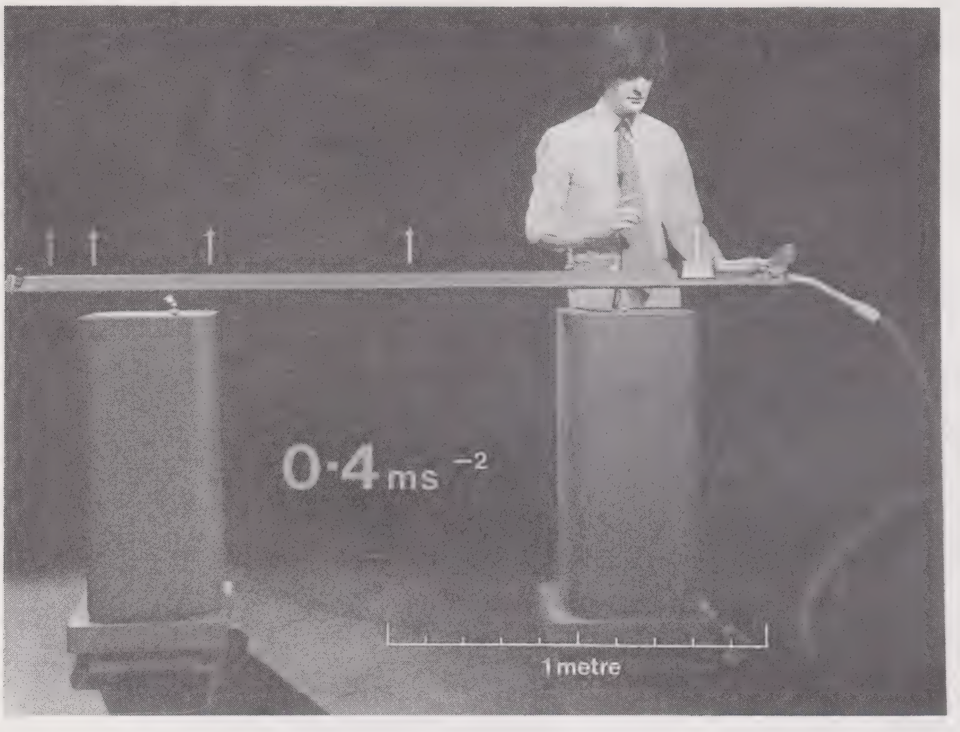


Figure 1

These experiments used the special piece of apparatus, known as an *air-track*, which is shown in Figure 1. This track allowed a continuous stream of air to be pumped upwards through tiny holes spaced along its surface. The function of the air-stream was to create a cushion which could support gliders above the surface of the track. The gliders were then able to move along the track with very little interference from friction, and this simplified the interpretation of the experiments. For most of the programme the track was kept strictly horizontal; this meant that gravity had no effect on the motion of the gliders.

Experiment 1

A standard glider was allowed to move freely along a horizontal air-track. Every $\frac{3}{4}$ second, its position was recorded by an after-image left on the screen. The after-images were equally spaced, indicating a constant velocity and zero acceleration.

Exercise 4

How can the equation $F = ma$ be used to explain this result?

[*Solution on page 51*]

Experiment 2

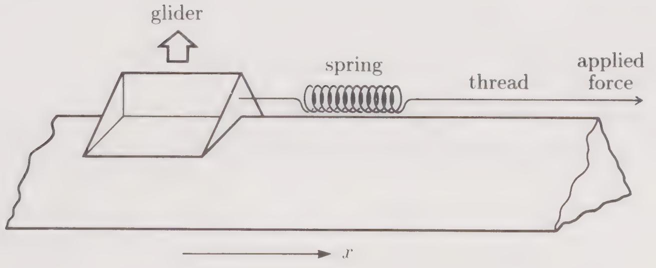


Figure 2

A standard glider was pulled along the track by a force transmitted through a piece of thread and a spring (Figure 2). The device pulling the thread was not shown or mentioned in the programme. This was deliberate, as it would be distracting to analyse the mechanisms that produce force—whether they are human muscles or electric motors. Instead, we accept that forces exist and concentrate on deducing their properties from the accelerations which they cause.

In Experiment 2, the glider was released from rest at $x = 0$ and $t = 0$. Its after-images, again taken at equal time intervals of $\frac{3}{4}$ second, spread further and further apart, indicating increasing speed. An analysis of this data led to the position-time graph of Figure 3(a) and to the velocity-time graph of Figure 3(b). These graphs showed that the glider had a constant acceleration of $0.2\,\mathrm{m\,s^{-2}}$.

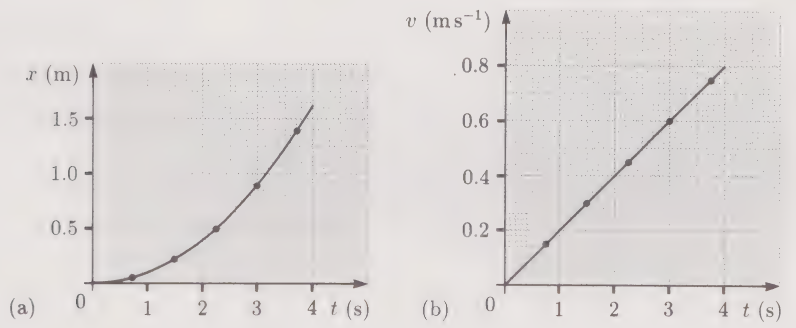


Figure 3

If $F = ma$ then a *constant* acceleration must be caused by a *constant* force. This prediction was checked by looking at the spring joining the thread to the glider. We made no detailed assumptions about the force needed to stretch the spring by a given amount. However, we did assume that any *change* in the force would result in a *change* of the spring's length. This allowed us to verify that the force was constant by checking that the length of the spring did not vary.

Experiment 3

A standard glider was released from rest at $x = 0$ and $t = 0$. It was pulled along the horizontal air-track by a force greater than before. The after-images, recorded every $\frac{3}{4}$ second starting from $t = 0$, are shown in Figure 1.

Exercise 5

(i) Taking measurements from Figure 1, with an x-axis pointing from left to right, plot a graph of position against time. (You should note that the photograph has been reduced so that the length scale can be conveniently used with a ruler graduated in *millimetres.*)

(ii) Draw tangents to the graph in part (i). Hence plot a graph of velocity against time.

(iii) Does your second graph agree (to within 10%) with the result quoted in the television programme ($a = 0.4\,\mathrm{m\,s^{-2}}$)?

[*Solution on page 51*]

If $F = ma$ then the acceleration of $0.4\,\mathrm{m\,s^{-2}}$ must have been caused by a constant force twice the strength of that used in Experiment 2 (where the acceleration was $0.2\,\mathrm{m\,s^{-2}}$). Again, this prediction was checked by looking at the springs joining the thread to the glider. This time two springs, each identical to that used in Experiment 2, were placed side by side (Figure 4). Each spring had the same extension as before; together, these extensions suggested twice as much force.

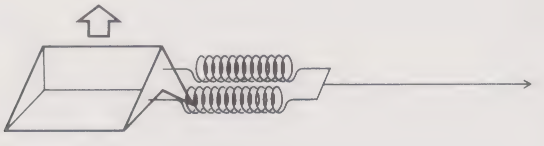


Figure 4

Experiment 4

Two standard gliders were stuck together and pulled along the horizontal air-track by the same force as that used in Experiment 2. The acceleration of this double glider was constant but was only $0.1\,\mathrm{m\,s^{-2}}$, half of that observed in Experiment 2. If $F = ma$ then the value of m, the proportionality constant in Newton's second law, must be twice as large for two gliders as for one.

Comments

1. The constant m is called the **inertial mass**, or **mass** for short. The word 'mass' is used because m is directly proportional to the amount of matter in the object (in our case, to the number of gliders stuck together). The word 'inertial' reminds us that m measures the inertia of an object, that is, its reluctance to change its velocity and deviate from unaccelerated motion.

2. So far the equation $F = ma$ has been used without mentioning how the numbers F and m are arrived at. Fortunately the equation *itself* tells us how this should be done. First we must adopt a convention: in the programme, a standard glider was defined to have a mass of 1 mass unit. This allowed us to quantify force by observing the acceleration of a standard glider. Thus in Experiment 2 the force has magnitude

$$1 \times 0.2 = 0.2 \text{ force units.}$$

In Experiment 3 the force has magnitude

$$1 \times 0.4 = 0.4 \text{ force units.}$$

The method for quantifying mass was illustrated with a non-standard glider of unknown mass. When Experiment 2 was repeated with the non-standard glider its acceleration was found to be $0.13\,\mathrm{m\,s^{-2}}$. Using the force calculated above for Experiment 2, the mass of the non-standard glider, which we denote here by m, was found to be

$$m = \frac{F}{a} = \frac{0.2}{0.13} \simeq 1.5 \text{ mass units.}$$

Exercise 6

Two gliders are subjected to the same constant force along a horizontal air-track. The first glider, of mass m, has an acceleration of $0.4\,\mathrm{m\,s^{-2}}$. The second glider, of unit mass, has an acceleration of $0.5\,\mathrm{m\,s^{-2}}$. What is the value of m?

[*Solution on page 51*]

Clearly any mass and any force can be quantified in this way, but Newton's second law also enables us to make predictions about the outcome of experiments.

Experiment 5

The air-track was tilted slightly and the non-standard glider released from rest. It accelerated down the slope at the constant rate of $0.06\,\mathrm{m\,s^{-2}}$. The force used in Experiment 2 was then used to pull the non-standard glider in the opposite direction. It accelerated up the slope at the constant rate of $0.07\,\mathrm{m\,s^{-2}}$.

In the first part of the experiment, there must have been a constant force acting down the track of

$$m \times 0.06 = 1.5 \times 0.06 = 0.09 \text{ force units.}$$

(This force is associated with gravity and the slope of the track, but these details are unimportant here.) In the second part of the experiment, an additional force of 0.2 force units was applied up the track. We assumed that 0.09 force units of this would cancel with the force acting down the slope, leaving a net or residual force of $0.2 - 0.09 = 0.11$ force units up the slope.

Exercise 7

Show that the value of the acceleration observed in the experiment could have been predicted using Newton's second law and a knowledge of the net force mentioned above.

[*Solution on page 51*]

Exercise 7 is mathematically simple, but even so it illustrates the power of Newton's theory to make predictions which agree with experiment.

Exercise 8

The non-standard glider of the television programme rests on an air-track which is tilted in exactly the same way as in Experiment 5. A thread is attached to the glider so that an additional force of magnitude F can be applied up the slope. Find the value of F under each of the following conditions:

(i) the glider is stationary;

(ii) the glider is moving uphill at a constant speed of $2\,\mathrm{m\,s^{-1}}$;

(iii) the glider is moving downhill at a constant speed of $2\,\mathrm{m\,s^{-1}}$;

(iv) the glider is moving downhill at a constant acceleration of $0.02\,\mathrm{m\,s^{-2}}$.

[*Solution on page 51*]

In the final part of the programme we considered the motion of an object falling through a fluid, that is, a liquid or gas. The fall of an object is caused by the gravitational force of attraction due to the Earth. The next experiment investigated how this force depends on the inertial mass of the object.

Exercise 9

An object hanging from a spring is subjected to two forces, a downward force due to the gravitational attraction of the Earth and an upward force caused by the extension of the spring (see Figure 5). If the object hangs at rest, what can you conclude about the magnitudes of these two forces?

[*Solution on page 51*]

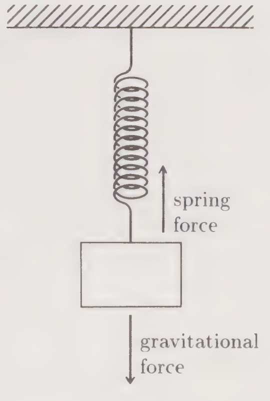


Figure 5

Experiment 6

This experiment illustrated the fact that the gravitational force of attraction on an object is proportional to the object's inertial mass. It was demonstrated that if a standard object extends a standard spring by a certain amount then two standard objects fixed together extend two standard springs by the same amount. Each of these two spring extensions represents an upward force of the same magnitude as that exerted by the original single spring. Hence the downward gravitational force on two standard objects is twice as large as that on one (see Figure 6 below). A similar result would be true for three standard objects and three springs, and so on.

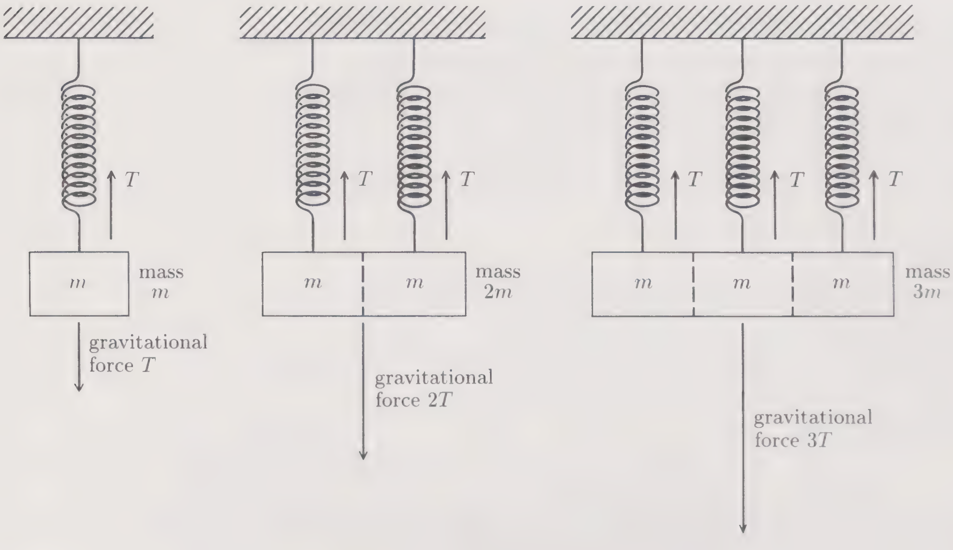


Figure 6

Hence the gravitational force on an object is proportional to its inertial mass. If the constant of proportionality is denoted by g then we have

$$\left.\begin{array}{l}\text{magnitude of the}\\ \text{gravitational force on}\\ \text{an object of mass } m\end{array}\right\} = mg.$$

The value of g varies slightly from point to point on the Earth's surface. The table on the right gives its sea-level value in SI units at four different places. The value of g in the vicinity of the United Kingdom is approximately $9.81\,\mathrm{m\,s^{-2}}$, and it is this value which we shall use from now on.

Place	$g\,(\mathrm{m\,s^{-2}})$
Equator	9.7810
London	9.8119
New York	9.8022
North Pole	9.8321

The final two experiments looked at simple models for bodies falling through a fluid. These models are outlined below, but will be discussed in much more detail in the next two sections.

Experiment 7

Consider a model for a falling object which assumes that the only force acting is the gravitational force of attraction due to the Earth (see Figure 7). Then Newton's second law predicts that the object will fall with constant downward acceleration of magnitude $g \simeq 9.81\,\mathrm{m\,s^{-2}}$. The model predicts further, using the result $x = \frac{1}{2}gt^2$ from Exercise 2, that the time of fall for an object from the Clifton Suspension Bridge (height 77 m) is 3.96 seconds. This compared well with the experimental value of 4.1 seconds. The reason for the slightly longer than predicted experimental time is that in actuality a second force, due to air resistance, also acts on the object.

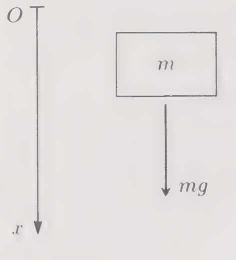


Figure 7

Experiment 8

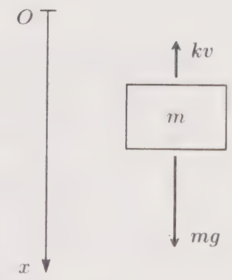


Figure 8

The previous model for a falling object can be improved by adding a resistive force, which for simplicity is assumed to be linearly proportional to the object's speed. As this force resists the motion, its direction is opposite to that of the velocity v and so is upwards. For this model, illustrated in Figure 8, Newton's second law gives

$$m\frac{dv}{dt} = mg - kv,$$

where k is a constant. A solution of this differential equation, as we asked you to verify in Exercise 3, is

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$

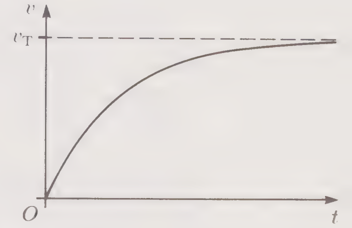


Figure 9

The graph of this velocity function is shown in Figure 9. The model predicts that the velocity will increase to the limiting value $v_T = mg/k$. This prediction was verified by observing the motion of a ball-bearing dropped into a bath of glycerine.

2.3 The fundamentals of Newtonian mechanics

To appreciate the significance of Newton's laws, you will need to know something of the ideas which they replaced. Before Newton, most scholars believed the teaching of Aristotle (384–322 BC). Aristotle's ideas seem plausible, but they turn out to be completely wrong. Aristotle's theory can be summarized by two slogans:

'The natural state of motion is rest', and

'Where there is motion, there must be a force'.

According to Aristotle, these slogans explain why you must push a cart to start it moving, and why you must keep on pushing if you want it to keep moving. In the absence of a force, motion across a horizontal surface was held to be impossible. That is why, Aristotle would say, the cart comes to a rather abrupt halt when you stop pushing it.

However, as you saw on the television programme, the glider moves with constant velocity in the absence of forces. You may also have seen television pictures of objects floating freely with constant velocity when released in a spacecraft. The reason why you need to exert a force on the cart in order to keep it moving with constant velocity is that there are internal frictional forces present (in the wheel bearings, for example). The force which you exert must cancel out these frictional forces before the cart can continue to move with constant velocity. A larger applied force will result in the cart accelerating. A similar situation arises in a car, where it is necessary to keep the accelerator pressed down to maintain a constant speed of 70 miles per hour along a horizontal road. In this case, as explained in the solution to Exercise 1(i), we need to balance the force of air resistance and also the frictional forces in the car's engine, transmission and wheel bearings.

Exercise 10

In order to maintain a car at constant speed on a horizontal road, it is necessary to apply a motive force by pressing down the accelerator pedal, as discussed above. Explain why the motive force required to maintain a speed of 70 mph is greater than that required to maintain a speed of 30 mph.

[*Solution on page 52*]

So far, the laws of Newtonian mechanics have been introduced quite informally, and their meaning illustrated for the special case of gliders on air-tracks. We shall now give precise statements of Newton's laws for the motion of any particle along a straight line. The justification for these laws lies in their success over a wide range of applications, some of which you will see later in the course.

Fundamental laws of Newtonian mechanics

Consider a particle moving along a straight line (the x-axis) whose origin and orientation are fixed.

The law of conservation of mass

The (inertial) mass of the particle is expressed by a single positive number, m. This number is an inherent property of the particle and does not depend on time, position, force or any other variable.

The law of addition of mass

If an object modelled by the particle is composed of a number of parts then the total mass m of the particle is the sum of the masses of the parts.

x-component of a force

The force acting on the particle at a given instant is represented by a single number, F, called the x-component of the force. This number may be zero, positive (if the force acts in the direction of increasing x) or negative (if the force acts in the direction of decreasing x). In general the x-component F of the force depends on time, position, velocity, etc., as the influences on the particle vary.

The law of addition of forces

If several forces act simultaneously on the particle then the x-component F of the net force or total force is the sum of the x-components of the individual forces. (This takes into account the directions of the individual forces as well as their magnitudes.)

Newton's second law

If the particle has mass m and experiences a force with x-component F then its acceleration a is given by

$$F = ma.$$

Newton's first law

When F is zero, a is zero: in the absence of a force the particle either stays permanently at rest or moves at constant velocity.

There are several points to be made about these laws. Note first that Newton's second law in the form $F = ma$ is applicable only to particles of constant mass. Thus this form of the law cannot be applied directly to rockets, which are losing mass through expelling exhaust gases, nor to raindrops, which gain mass through the condensation of water vapour upon them or lose mass via evaporation.

The equation $F = ma$ assumes that force and mass are measured in a very definite way. Firstly, to measure mass m, a reference object must be selected. In the television programme we used a standard glider as our reference and specified that it had a mass of one unit. From now on we shall adopt the SI convention which takes as its reference a cylinder of platinum-iridium alloy kept at the Bureau of Metric Standards, Sèvres, France, and known as the *standard kilogram.*

Mass and force are then quantified in much the same way as shown in the television programme. The number m, representing the mass of a particle, is measured by the acceleration of the standard kilogram divided by the acceleration of the particle, when both are subjected to the same force (see Exercise 6). Masses quantified in this way are said to be expressed in *kilograms.* The number F, representing a force, is measured by the acceleration (in $\mathrm{m\,s^{-2}}$) given to the standard kilogram. Forces quantified in this way are said to be expressed in *newtons.* Thus, a force of 1 newton causes a mass of 1 kilogram to accelerate at $1\,\mathrm{m\,s^{-2}}$, or a mass of 2 kilograms to accelerate at $\frac{1}{2}\,\mathrm{m\,s^{-2}}$.

The SI unit for mass is the kilogram, abbreviated as kg.

The SI unit for force is the newton, abbreviated as N. $1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$.

As pointed out above, the x-component F of a force may be zero, positive or negative. If $F = 0$ at some moment then no net force is acting on the particle. Otherwise the force acting will have both a magnitude and a direction. The magnitude, measured in newtons, is the positive number $|F|$ (or zero in the case when no force acts). For the direction there are just two possibilities, since here we are considering only one-dimensional motion: the force may act either in the direction of increasing x (as indicated by the arrowhead on one end of the x-axis) or in the direction of decreasing x. The force's x-component F will be positive in the first case and negative in the second (see Figure 10). Thus, for example, the x-component $2\,\mathrm{N}$ represents a force of magnitude 2 newtons acting in the direction of increasing x, whereas the x-component $-3\,\mathrm{N}$ represents a force of magnitude 3 newtons acting in the direction of decreasing x.

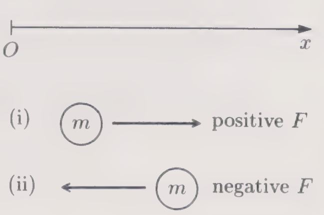


Figure 10

We now have three equivalent ways of describing a force, each of which takes into account both magnitude and direction.

1. An arrow on a diagram, with a positive number or symbol beside the arrowhead. Here the arrow represents the direction of the force, and the number or symbol stands for its magnitude (see Figures 7 and 8, for example).
2. A verbal description, such as 'a force of magnitude mg acting downwards' or 'a force of magnitude kv acting upwards'.
3. An algebraic description, using the x-component F. Here the force has

   magnitude $|F|$

   and acts

   $$\text{in the direction of } \begin{cases} \text{increasing } x & \text{if } F > 0, \\ \text{decreasing } x & \text{if } F < 0. \end{cases}$$

The algebraic description is dependent on the prior choice of an orientation for the x-axis, and it is for this reason that we refer to the number F as the *x-component* of the force. Sometimes this is emphasized by writing F_x rather than just F for the force x-component, though we shall not do so in this unit.

Since it is rather a mouthful to speak of 'the force with x-component F', this is often abbreviated to 'the force F'. Note that it makes sense to talk of 'the force F' only when the direction of the x-axis has been chosen beforehand, and the sign of F depends on this choice. (The same is true of velocity v and acceleration a, as explained in Section 1.)

It is clear from the equation $F = ma$ that at any instant the sign of the acceleration a is the same as that of the force F, since the mass m of a particle is a positive number. This is as expected; a force in a particular direction causes acceleration in that same direction.

If more than one force acts on a particle then the number F appearing in Newton's second law $F = ma$ is the (x-component of the) net force or total force acting on the particle. This is calculated from the individual forces which act by taking their algebraic sum, that is, the sum of their x-components. You saw some examples of the combination of forces in the television programme. Experiment 3, for example, involved a glider being accelerated by two equal forces of magnitude 0.2 force units. On choosing an x-axis in the direction of the motion, as shown in Figure 11, the two forces each have x-component 0.2 units and the x-component of the net force is $0.2 + 0.2 = 0.4$ units.

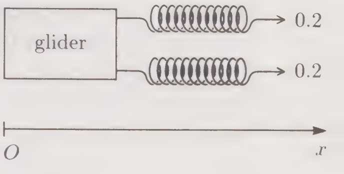


Figure 11

In Experiment 5 the glider was subjected to a gravitational force down the sloping air-track of magnitude 0.09 force units and a pulling force up the slope of magnitude 0.2 units. With an x-axis pointing up the slope, as in Figure 12, these forces have respective x-components -0.09 units and 0.2 units, so that the x-component of the net force is $0.2 + (-0.09) = 0.11$ units (that is, magnitude 0.11 units and direction up the slope).

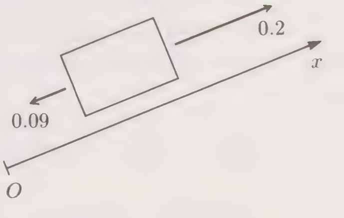


Figure 12

Finally, in Experiment 8, the ball-bearing was subjected to the gravitational force of magnitude mg downwards and a resistive force of magnitude kv upwards. Using an x-axis pointing downwards (see Figure 13), the net force F is the sum of the x-components of the two individual forces, that is,

$$F = mg + (-kv) = mg - kv.$$

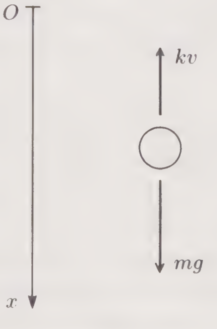


Figure 13

Adding the x-components of the individual forces which act on a particle is equivalent to

adding the magnitudes of those forces in the direction of increasing x

and then

subtracting the magnitudes of those forces in the direction of decreasing x.

This is therefore another way of thinking about finding the net force. Figure 14 illustrates how the x-component F of the net force is obtained from the magnitudes A, B, C and D of four individual forces (which have respective x-components A, B, $-C$ and $-D$).

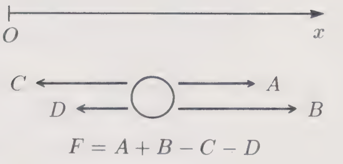


Figure 14

Exercise 11

The diagrams in Figure 15 below each indicate the individual forces acting on a particle and the direction of increasing x. In each case, specify the force F which should appear in Newton's second law $F = ma$.

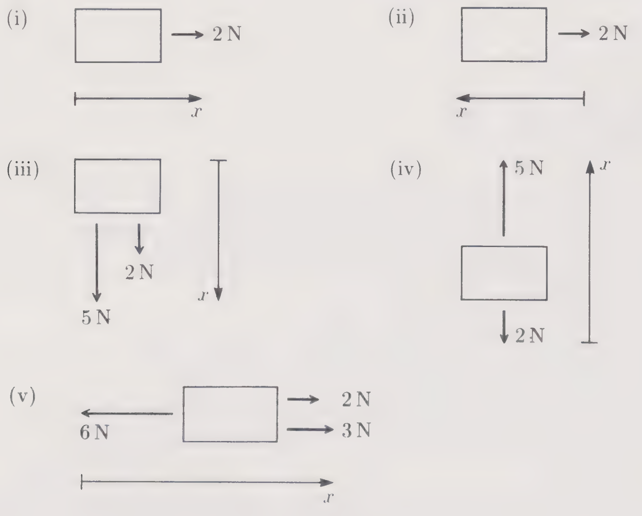


Figure 15

[*Solution on page 52*]

When forces are combined their directions have to be taken into account as well as their magnitudes. For one-dimensional motion this combination process may be dealt with using ordinary addition or subtraction, as you have seen. A different approach is required when we consider forces in two- or three-dimensional situations, where the forces need not all be directed along the same straight line. In this case we model forces (as well as positions, velocities and accelerations) by *vectors*, which are mathematical objects having both magnitude and direction. The combination of several forces is achieved by *vector addition.* The algebra of vectors is introduced in *Unit 14*, and its application to three-dimensional mechanics in *Unit 15.* For the current study of one-dimensional motion the vector model is not essential, but the mathematics involved here will be seen later to correspond to that of an individual component in the three-dimensional vector situation.

2.4 Newton's first law

The way in which we have introduced Newton's laws of motion may give the impression that the first law is a special case of the second law and therefore redundant. However, this is not the case; the first law is essential to the definition of force. It allows us to define a suitable set of axes (or a single axis in the one-dimensional case) with respect to which Newton's laws apply.

You may have realized that our definition of force is somewhat arbitrary, in that it depends on measuring the acceleration of a known mass using a particular set of coordinate axes. If we chose instead to use a second set of axes, which is accelerating relative to the first, then we would obtain a different answer for the magnitude and direction of the force! Newton believed in the existence of a frame of reference which is 'absolutely at rest' and within which his laws would apply. It is now thought that the search for such an absolute frame of reference is fruitless. However, in theory at least, we can consider a particle in deep space, far from any other matter. There will be residual forces acting on this particle because, for example, all matter has a gravitational attraction on all other matter, no matter how distant, but these forces will be negligible if the particle is sufficiently removed from all other objects. Then, by

Newton's first law, the particle will be at rest or moving with constant velocity. We can use this particle's motion to define a suitable set of coordinate axes with respect to which Newton's laws apply. For practical purposes it is found that a set of axes which is 'fixed relative to the fixed stars' is sufficient. If a particle is accelerating relative to these axes then the first law tells us that some agency must be causing the acceleration and the second law allows us to measure the force involved.

The 'fixed stars' are those distant stars which have no discernible lateral movement when viewed from the Earth.

In this course we generally use a set of axes which is fixed on the Earth's surface. This frame is actually accelerating, due mainly to the Earth's rotation. The effects of this acceleration are small in most cases, though small effects acting for long times can have important consequences. For example, it would be necessary to take the rotation of the Earth into account in order to calculate accurately the trajectory of an inter-continental missile. On a smaller scale, the Earth's rotation causes an object dropped from a height of 100 m in the United Kingdom to diverge from the vertical by about 2 cm to the East. In applying Newton's laws to a set of axes fixed on the Earth's surface we are ignoring such small effects as a part of the modelling process.

Effects due to the wind are liable to be larger than this.

Newton's first law is also the basis of the subject of *statics*, which deals with the equilibrium of systems at rest. It tells us that if an object is at rest then the total force acting on it is zero. The application of this principle to one-dimensional systems is relatively straightforward, as you saw in Exercise 9. When an object hangs at rest from a spring, there must be an upward force due to the spring whose magnitude equals that of the downward gravitational force on the object (see Figure 16). The equilibrium of three-dimensional systems will be considered later in the course.

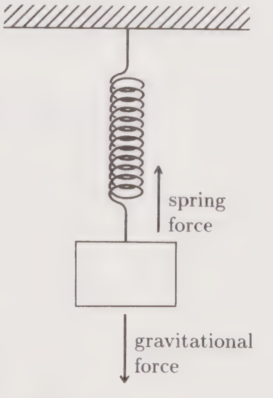


Figure 16

Exercise 12

In each of the following cases, identify the forces acting on the object, stating their magnitudes and directions:

(i) a stationary object of mass m hung on a string;

(ii) a stationary object of mass m on a horizontal table.

[*Solution on page 52*]

Summary of Section 2

1. **Newton's first law** states that, in the absence of a force, a particle either stays permanently at rest or moves at constant velocity.

2. The **force** acting on a particle at a given instant is represented by a single number, F, called the force's ***x*-component**. The force has magnitude $|F|$, and is in the direction of increasing x if $F > 0$ or in the direction of decreasing x if $F < 0$. Often 'the force with x-component F' is shortened to 'the force F'.

3. **Newton's second law** states that if a particle of mass m experiences a net force with x-component F then its acceleration a is given by $F = ma$.

4. If more than one force acts on a particle, then the **net force** or **total force** F to be used in Newton's second law is the algebraic sum of the individual forces, that is, the sum of their x-components.

3 First model for the motion of a falling object (Audio-tape Section)

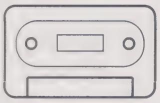
3.1 Introduction

The first two sections of this unit outlined the framework of Newtonian mechanics. The basic quantities of position, velocity, acceleration, force and mass were introduced, and the relationships between them explored. The next two sections contain a selection of examples and exercises based on the two models for the motion of a falling object which were used in Experiments 7 and 8 of the television programme. You should attempt as many of the exercises as you can, checking your answers against the solutions at the end of the units. The general techniques of solution developed here apply to many of the mechanics problems to be met later in this course.

However, before examining these two models we shall consider general strategies for integrating the differential equation which arises from Newton's second law of motion.

3.2 Integrating Newton's second law

Start the audio-tape when you are ready.



Tape Section 1

1 Newton's Second Law

Force = mass x acceleration

$$\boxed{F = ma}$$

GIVEN F $\Longrightarrow$ FIND motion

$$a = \frac{F}{m}$$

$$F = F(t, x, v)$$

2 Acceleration

Definition $\quad a = \dfrac{dv}{dt} \qquad$ (i)

OR $\quad a = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2} \qquad$ (ii)

OR $\quad a = \dfrac{dx}{dt}\,\dfrac{dv}{dx} = v\dfrac{dv}{dx} \qquad$ (iii)

3 Case 1 : $F = F(t)$

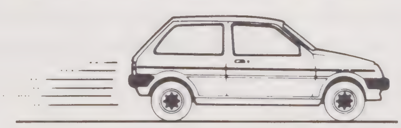
$$a = \frac{F}{m} = \text{function of } t \text{ only} = f_1(t).$$

Use $a = \dfrac{dv}{dt}$ $\Longrightarrow$ $\dfrac{dv}{dt} = f_1(t)$

$\Longrightarrow$ $v = \displaystyle\int f_1(t)\,dt + c$

Use initial condition to find this

4 Example 1 : $F(t) = mkt$



e.g. the force propelling a car forward initially if the accelerator pedal is pressed down increasingly.

- Car starts from rest at the origin at $t = 0$.

$a = \dfrac{F}{m}$ $\Longrightarrow$ $a = \boxed{\quad}$ function of t only

Use $a = \dfrac{dv}{dt}$ $\Longrightarrow$ $\dfrac{dv}{dt} = \boxed{\quad}$

Integrate: $\quad v = \boxed{\quad} + c_1$

'starts from rest' $\quad v = 0$ at $t = 0$: $\quad c_1 = \boxed{\quad}$ Use initial condition

Therefore $v(t) = \boxed{\quad}$.

5 Example 1 – continued

$$v(t) = \frac{1}{2}kt^2$$

Use $v = \dfrac{dx}{dt}$ $\Longrightarrow$ $\dfrac{dx}{dt} = \boxed{\quad}$

Integrate: $x = \boxed{\quad} + c_2$

'starts ... at the origin' $x = 0$ at $t = 0$: $c_2 = \boxed{\quad}$

Use initial condition

Therefore $x(t) = \boxed{\quad}$

6 Case 2: $F = F(x)$

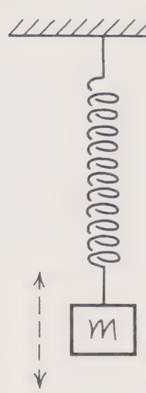
$$a = \frac{F}{m} = \text{function of } x \text{ only} = f_2(x).$$

Use $a = v\dfrac{dv}{dx}$ $\Longrightarrow$ $v\dfrac{dv}{dx} = f_2(x)$

Separation of variables $\Longrightarrow$ $\displaystyle\int v\,dv = \int f_2(x)\,dx + c$

Integrate: $\dfrac{1}{2}v^2 = \displaystyle\int f_2(x)\,dx + c.$

7 Example 2: $F(x) = mg - kx$



e.g. the force acting downwards on an object of mass m suspended from a spring whose extension is x.

$a = \dfrac{F}{m}$ $\Longrightarrow$ $a = \boxed{\quad}$ function of x only

Use $a = v\dfrac{dv}{dx}$ $\Longrightarrow$ $v\dfrac{dv}{dx} = \boxed{\quad}$

Separate variables: $\displaystyle\int \boxed{\quad}\,dv = \int\left(\boxed{\quad}\right)dx + c$

Integrate: $\boxed{\quad} = \boxed{\quad} + c$

• Assume $v = v_0$ at $x = 0$ $\Longrightarrow$ $c = \boxed{\quad}$ Use initial condition

Therefore $\boxed{\quad} = \boxed{\quad}$

8 Case 3: $F = F(v)$

$$a = \frac{F}{m} = \text{function of } v \text{ only} = f_3(v).$$

TWO APPROACHES

(i) Use $a = \dfrac{dv}{dt}$

$$\Longrightarrow \frac{dv}{dt} = f_3(v)$$

$$\Longrightarrow \int 1\,dt = \int \frac{1}{f_3(v)}\,dv + c$$

$$\text{or} \quad t = \int \frac{1}{f_3(v)}\,dv + c$$

Obtain v in terms of t

(ii) Use $a = v\dfrac{dv}{dx}$

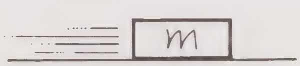
$$\Longrightarrow v\frac{dv}{dx} = f_3(v)$$

$$\Longrightarrow \int 1\,dx = \int \frac{v}{f_3(v)}\,dv + c$$

$$\text{or} \quad x = \int \frac{v}{f_3(v)}\,dv + c$$

Obtain v in terms of x

9 Example 3 (i): $F(v) = -mkv$



e.g. linear resistance force on ice puck.

- Projected from origin at $t=0$ with initial velocity v_0.

$$a = \frac{F}{m} \Longrightarrow a = \boxed{}$$

function of v only

Use $a = \dfrac{dv}{dt}$ $\Longrightarrow$ $\dfrac{dv}{dt} = \boxed{}$

Separate: $\displaystyle\int \boxed{}\,dv = \int \left(\boxed{}\right) dt + c_1$

Integrate: $\boxed{} = \boxed{} + c_1$

$v = v_0$ at $t = 0$: $c_1 = \boxed{}$

Use initial condition

So $\boxed{} = \boxed{} + \boxed{}$

Therefore $v(t) = \boxed{}$.

10 Example 3(i) – continued

$v(t) = v_0 e^{-kt}$

Use $v = \dfrac{dx}{dt}$ $\Longrightarrow$ $\dfrac{dx}{dt} = \boxed{}$

Integrate: $x = \boxed{} + c_2$

$x = 0$ at $t = 0$: $c_2 = \boxed{}$ Use initial condition

Therefore $x(t) = \boxed{}$.

11 Example 3(ii): $F(v) = -mkv$

$a = \dfrac{F}{m}$ $\Longrightarrow$ $a = \boxed{}$ Function of v only

Use $a = v\dfrac{dv}{dx}$ $\Longrightarrow$ $v\dfrac{dv}{dx} = \boxed{}$

or $\dfrac{dv}{dx} = \boxed{}$

Integrate: $v = \boxed{} + c$

$v = v_0$ at $x = 0$: $c = \boxed{}$ Use initial condition

Therefore $v(x) = \boxed{}$.

12 Summary

	Form of force	Use	Obtain
1	$F = F(t)$	$a = \dfrac{dv}{dt}$	v in terms of t
2	$F = F(x)$	$a = v\dfrac{dv}{dx}$	v in terms of x
3	$F = F(v)$	either $a = \dfrac{dv}{dt}$	v in terms of t
		or $a = v\dfrac{dv}{dx}$	v in terms of x

Exercise 1

A particle of mass m starts from rest ($v = 0$) at the origin ($x = 0$) at time $t = 0$, and is then subjected to a force with x-component F. Find the particle's velocity v and position x as functions of time t in each of the following cases (k and ω are constants).

(i) $F = mkt^2$

(ii) $F = k \sin \omega t$

Exercise 2

A particle of mass m starts from rest at the origin, and is then subjected to a force with x-component F. Find the particle's velocity v as a function of position x in each of the following cases (k, ω and b are positive constants).

(i) $F = k \cos \omega x$

(ii) $F = ke^{bx}$

Exercise 3

An ice-puck of mass m sliding on a smooth horizontal surface moves in the direction of increasing x and experiences a quadratic air resistance force $F = -kv^2$, where k is a positive constant. Initially, at $t = 0$, the puck is at the origin and has velocity v_0. By integrating the equation of motion (that is, Newton's second law),

(i) find the velocity v and position x of the puck as functions of time t;

(ii) find the velocity v as a function of position x.

Exercise 4

In each of the following cases, suggest an expression for the acceleration a which turns Newton's second law $F = ma$ into a differential equation soluble in principle by integration (that is, containing just two of the variables x, v and t).

(i) The force is a function of both time and velocity, that is,

$$F = F(t, v).$$

(ii) The force is a function of both position and velocity, that is,

$$F = F(x, v).$$

[*Solutions on page 52*]

3.3 Motion under gravity

We return now to considering the motion of falling objects. The basic reason why free bodies fall is the gravitational force of attraction due to the Earth. Indeed this force acts on *all* bodies, whether they are falling, stationary or in any type of constrained motion. In the television programme we investigated this force by suspending standard masses from standard springs, and demonstrated the following important results.

The force of gravity

Every object near the Earth's surface is pulled vertically downwards by the gravitational attraction of the Earth. The magnitude of this force is independent of the object's position and is proportional to its inertial mass m, that is,

$$\text{magnitude of force of gravity } = mg,$$

where the constant of proportionality g is approximately $9.81\,\mathrm{m\,s^{-2}}$ in SI units.

The fact that the force of gravity is proportional to the inertial mass is extremely important. For example, it means that we can use weighing scales to ascertain the mass of a body, rather than the dynamic process of measuring its acceleration when subjected to a known force. Strictly speaking the *weight* of an object is the magnitude of the force of gravity acting on the body, and should be measured in newtons. However, your bathroom scales are tabulated in kilograms (or the equivalent imperial units of stones and pounds), which are the units of mass. In everyday speech we generally employ the

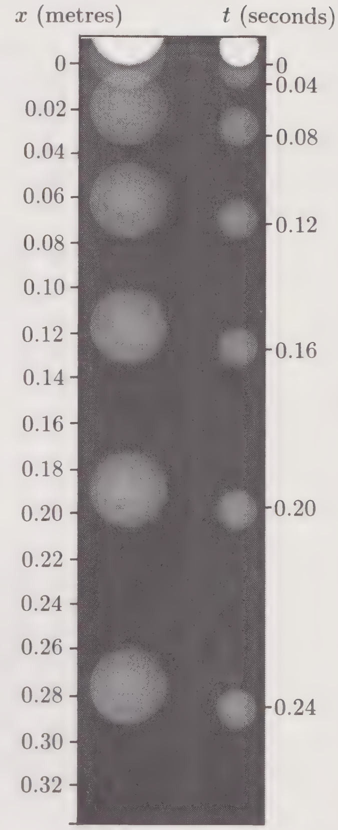


Figure 1
After-images of two balls of differing sizes falling through air. The results are slightly inaccurate because of shortcomings in the magnetically based method of releasing the balls. (This figure is referred to on the next page of the text.)

Figure 2

Note that the force of gravity has a constant magnitude (approximately) only for points near the Earth's surface. For heights greater than about 1 km a more exact expression is required, as you will see in *Unit 29*.

word 'weight' when we should be using 'mass'. The sense of the word 'weight' should be clear from its units: if it is measured in kilograms then it means 'mass'; if it is measured in newtons then it means 'the magnitude of the force of gravity'.

One way of checking this law for the force of gravity is to observe the motion of falling objects. If gravity is the only force acting, Newton's second law predicts a downward acceleration of magnitude $F/m = g \simeq 9.81\,\mathrm{m\,s^{-2}}$. In a vacuum near the Earth's surface any falling object is expected to have this acceleration, regardless of its mass. However, the presence of air may modify this prediction because another force, the force of air resistance, comes into play. Sometimes, the effects of air resistance are vital (as in the case of a parachute), but often they are small and can be neglected.

For this reason the constant of proportionality g is often called the *magnitude of the acceleration due to gravity.*

This is the case in the experiment illustrated in Figure 1 on the previous page, in which two balls of differing sizes are released from rest. Both balls start with their lowest points aligned. You can see that this alignment is preserved almost exactly as the balls fall through the air. Mathematical analysis further confirms a common constant acceleration of approximately $9.81\,\mathrm{m\,s^{-2}}$, as predicted using the gravitational force law above.

Figure 3

Figure 4

We shall now use this simple model, in which we assume that gravity is the only force acting, to make some predictions about falling objects. Before asking you to look at some examples and exercises on this theme, we summarize the general procedure for applying Newton's second law to particles in straight-line motion. Figures 3–5 and the following formulas in the margin illustrate this procedure in action.

Procedure for applying Newton's second law to straight-line motion

1. Draw a dot to represent the *particle.* Write m on the diagram to indicate its mass (Figure 3).
2. For each *force* acting on the particle, draw an arrow in the corresponding direction. Write the magnitude of the force beside the arrow. (If you do not know the precise value of the magnitude then use a symbol to represent it, as in Figure 4.)
3. Choose an *x-axis* and mark it on the diagram (Figure 5).
4. Compute the value of F, the *x-component of the total force* acting on the particle, by taking the algebraic sum of the individual forces.
5. Substitute this value into *Newton's second law*, $F = ma$, to give the equation of motion. This is the starting point for the mathematical analysis, which will usually involve solving a differential equation.

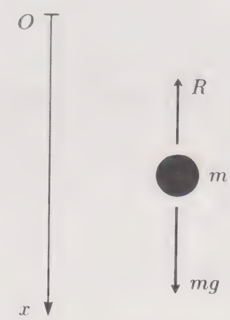


Figure 5

$F = mg - R$

$ma = mg - R$

Notes on procedure

1. The importance of drawing a diagram cannot be stressed too strongly.
2. In some cases you may not know the direction of the force in advance. If this is the case then make an informed guess. If your guess is incorrect then the symbol chosen to represent the 'magnitude' of the force will turn out to have a negative value, indicating that the direction of the force is in fact opposite to that shown on your diagram.
3. The choices of the origin and the direction of the x-axis are arbitrary, and will have no significant effect on the final outcome of your calculations. It is simplest if possible to make these choices so that the position x and/or the velocity v are positive for the particle's motion, but this is not essential.

To illustrate this procedure, you should now read through the following example before listening to the audio-tape.

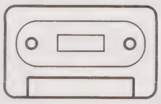
Example 1

An object, which is initially at rest is dropped from the Clifton Suspension Bridge and falls into the River Avon, 77.0 metres below. Assuming that the only force acting on the object is the force of gravity, find

(i) the time of fall;

(ii) the speed of the object just before it hits the water.

Start the audio-tape when you are ready.



Tape Section 2

Working Notes for Example 1

1. Forces acting on object: gravity mg downwards.

2. Initial conditions: $v = 0$ and $x = 0$ when $t = 0$.

3. What are we asked to find?

   (i) Value of t when $x = 77.0$

   (ii) Value of v when $x = 77.0$

4. Strategy for solution:

   (i) apply Newton's Second Law $F = ma$;

   (ii) use $a = \dfrac{dv}{dt}$ and integrate to find v in terms of t;

   (iii) use $v = \dfrac{dx}{dt}$ and integrate to find x in terms of t;

   (iv) find time t when $x = 77.0$, then find v at this time.

Solution to Example 1

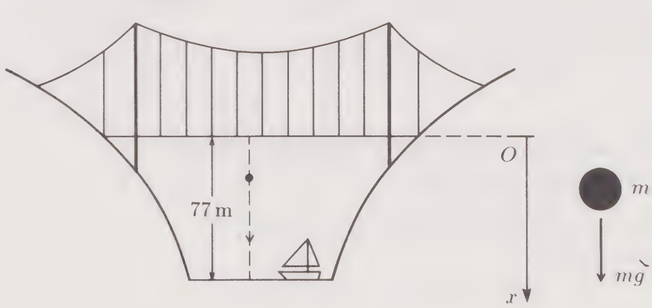


Figure 6

The only force acting on the object is assumed to be gravity, of magnitude mg downwards. We choose an x-axis pointing vertically downwards with its origin at the point where the object is released.

Putting $F = mg$ into Newton's second law $F = ma$ gives

$$ma = mg,$$

that is, $a = g$. Hence

$$\frac{dv}{dt} = g.$$

On integrating, we obtain

$$v = gt + c_1,$$

where c_1 is a constant. Now $v = 0$ when $t = 0$, so $c_1 = 0$ and we have

$$v = gt. \tag{1}$$

With dx/dt in place of v this is

$$\frac{dx}{dt} = gt,$$

leading on integration to

$$x = \tfrac{1}{2}gt^2 + c_2.$$

The initial condition states that $x = 0$ when $t = 0$, so $c_2 = 0$ and

$$x = \tfrac{1}{2}gt^2. \tag{2}$$

(i) When $x = 77.0$ we have

$$77.0 = \tfrac{1}{2}gt^2,$$

which on putting $g = 9.81$ gives

$$t = \sqrt{\frac{2 \times 77.0}{9.81}} = 3.962.$$

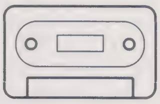
The object therefore hits the water 3.96 s after being released.

(ii) Putting $t = 3.962$ into Equation (1) produces

$$v = 9.81 \times 3.962 = 38.87,$$

so the object has a speed of $38.9\,\mathrm{m\,s^{-1}}$ just before it hits the water. □

Restart the audio-tape when you are ready.



General scheme for solution

1. Visualize the physical situation, drawing a diagram if necessary.
2. Draw a symbolic diagram, indicating the physical system, the forces acting on it and the coordinate axis.
3. Read the question again, noting any information which might be useful.
4. Note what the question asks, and formulate a strategy to answer it.
5. Execute your strategy.
6. Perform any readily available checks on your working, and consider whether your answers are physically reasonable.
7. Read the question once more as a final check that you have fully answered the question which was set.

Example 2

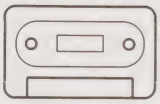
A particle is thrown vertically upwards from ground level with an initial speed u. Find

(i) the time taken for the particle to reach its maximum height;

(ii) the maximum height attained;

(iii) the time taken for the particle to return to the ground;

(iv) the speed of the particle as it reaches the ground on its return.



Restart the audio-tape when you are ready.

Working Notes for Example 2

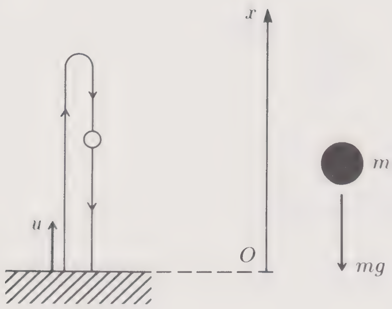
1. Forces acting on object?

2. Initial conditions?

3. What are we asked to find?

4. Strategy for solution?

Solution to Example 2



For clarity, the diagram distinguishes the upward and downward phases of the particle's motion. In fact these both take place along the same vertical line.

Figure 7

We assume that the only force acting on the body is gravity, of magnitude mg downwards. We choose an x-axis pointing vertically upwards, with its origin at the point of projection.

The x-component of the total force is $F = -mg$, and substituting this into Newton's second law $F = ma$ gives

$$ma = -mg,$$

that is, $a = -g$. Hence

$$\frac{dv}{dt} = -g.$$

Integration produces

$$v = -gt + c_1.$$

Now $v = u$ when $t = 0$, so that $c_1 = u$ and

$$v = u - gt. \tag{3}$$

Writing this as

$$\frac{dx}{dt} = u - gt$$

and integrating leads to

$$x = ut - \tfrac{1}{2}gt^2 + c_2.$$

Since $x = 0$ when $t = 0$ we find that $c_2 = 0$ and

$$x = ut - \tfrac{1}{2}gt^2. \tag{4}$$

(i) The particle reaches its maximum height when $v = 0$, and from Equation (3) this occurs at time

$$t = \frac{u}{g}.$$

Since the motion started at $t = 0$, it takes a time u/g for the particle to reach its maximum height.

As a check, the units of u/g are

$$\frac{\mathrm{m\,s^{-1}}}{\mathrm{m\,s^{-2}}} = \mathrm{s},$$

verifying that the answer produced is in seconds.

(ii) Substituting $t = u/g$ into Equation (4), we have

$$x = \frac{u^2}{g} - \frac{u^2}{2g} = \frac{u^2}{2g},$$

so the maximum height attained by the particle is $u^2/(2g)$.

(iii) The particle reaches the ground when $x = 0$, and from Equation (4) this occurs when

$$ut - \tfrac{1}{2}gt^2 = 0$$

or $$t(u - \tfrac{1}{2}gt) = 0.$$

Hence $t = 0$ or $t = 2u/g$. Now $t = 0$ corresponds to the time of projection, so the particle returns to the ground after a time $2u/g$.

(iv) Substituting $t = 2u/g$ into Equation (3) gives

$$v = u - 2u = -u,$$

so the particle reaches the ground on its return with speed u. □

Exercise 5

A stone, dropped from rest, takes 3.00 seconds to reach the bottom of a well. Assuming that gravity is the only force acting on the stone, estimate

(i) the depth of the well;

(ii) the speed of the stone when it reaches the bottom.

Exercise 6

A man leaning from a window throws a ball vertically upwards from a point 4.40 m above the ground. The initial speed of the ball is $7.60\,\mathrm{m\,s^{-1}}$. It travels up and then down in a straight vertical line and eventually reaches the ground. Assuming that gravity is the only force acting on the ball, estimate

(i) the time that elapses before the ball reaches the ground;

(ii) the speed of the ball when it strikes the ground.

Exercise 7 (Example 1 revisited)

An object which is initially at rest is dropped from the Clifton Suspension Bridge and falls into the River Avon, 77.0 metres below. Assume that the only force acting on the falling object is the force of gravity.

(i) By putting $a = v\,dv/dx$ in Newton's second law and solving the resulting differential equation, find the object's velocity v as a function of the distance x through which the object has fallen.

(ii) By putting $v = dx/dt$ in the result of part (i), find the time t which the object takes to fall a distance x.

(iii) Hence find the total time of fall and the speed of the object just before it hits the water.

[*Solutions on page 53*]

Summary of Section 3

In this section Newton's second law has been used to model the motion of an object falling under the force of gravity, where the effects of air resistance are ignored.

1. Newton's second law $F = ma$ may in principle be integrated by putting

(i) $a = \dfrac{dv}{dt}$ (giving v in terms of t) if $F = F(t)$ or $F = F(v)$;

(ii) $a = v\dfrac{dv}{dx}$ (giving v in terms of x) if $F = F(x)$ or $F = F(v)$.

2. Any object near the Earth's surface is pulled downwards by the **force of gravity**. The magnitude of this force is proportional to the object's inertial mass m, that is,

$$\text{force of gravity} = mg \text{ downwards},$$

where the constant of proportionality g is approximately $9.81\,\mathrm{m\,s^{-2}}$.

4 Second model for the motion of a falling object (Audio-tape Section)

4.1 Air resistance

In many situations our first model for a falling object, which took only the force of gravity into account, is perfectly adequate. For example, it predicted a time of fall from the Clifton Suspension Bridge of 3.96 seconds, which was close to the experimental value of 4.1 seconds. The slight difference between prediction and measurement here is due to air resistance, and if we had dropped a basketball rather than a piece of metal tubing then the discrepancy would have been more significant.

The idea of air resistance is quite familiar in everyday life. For example, any cyclist knows the following.

1. Air resistance tends to slow one down and resists attempts by the cyclist to increase speed.
2. At low speeds air resistance has little effect but at higher speeds it becomes more noticeable, making it difficult to cycle faster than about 40 km per hour.
3. Air resistance can be reduced by crouching over the handlebars to present a smaller profile to the wind.

From these observations we conclude that air resistance is a force whose direction is opposite to that of the motion and whose magnitude, R, depends on the object's speed, shape and size. We expect that this magnitude will increase from zero as the object's speed $|v|$ increases, but the precise nature of the dependence can be discovered only by experiment. In most cases a reasonable approximation is found to be

$$R \simeq k_1|v| + k_2|v|^2,$$

where k_1 and k_2 are constants which depend on the shape and size of the object.

Using this expression for R in Newton's second law leads to differential equations which are somewhat complicated to solve. However, for small speeds the linear term dominates, so that here we can use the approximation

$$R \simeq k_1|v|,$$

whereas for large speeds the quadratic term dominates, so that we can then use the approximation

$$R \simeq k_2|v|^2.$$

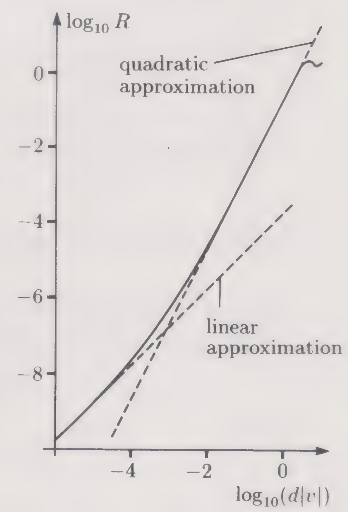


Figure 1
A graph of $\log_{10}R$ against $\log_{10}(d|v|)$, where R is the magnitude of the force of air resistance (in newtons) opposing the motion of a sphere of diameter d metres moving at a speed of $|v|\,\mathrm{m\,s^{-1}}$.

Each of these approximations leads to more amenable differential equations. In Figure 1 we show in miniature typical experimental results for smooth spheres of diameter d and speed $|v|$. The experiments show that R depends on the product of d and $|v|$ so that, for example, a sphere of diameter 1 metre and speed $1\,\mathrm{m\,s^{-1}}$ experiences the same air resistance as a sphere of diameter 2 metres and speed $0.5\,\mathrm{m\,s^{-1}}$. We have therefore plotted the values of R against $d|v|$. Because of the enormous range of values involved, a log-log plot has been used. The best fit of the form $R = k_1|v| + k_2|v|^2$ to the experimental data is given by $k_1 = 1.7 \times 10^{-4}d$ and $k_2 = 0.20d^2$ in SI units. We also find that the linear and quadratic expressions can be used over the following ranges:

$$R \simeq 1.7 \times 10^{-4}d|v| \qquad \text{for } d|v| \lesssim 10^{-5},$$
$$R \simeq 0.20d^2|v|^2 \qquad \text{for } 10^{-2} \lesssim d|v| \lesssim 1.$$

The symbol $\lesssim$ means 'less than about ...'.

4.2 Motion under gravity and air resistance

The linear and quadratic approximations introduced in the previous subsection allow us to improve the modelling undertaken in Section 3 for motion under gravity. The problems of this subsection continue the theme of falling objects, but now we take into account the effects of air resistance. From a physical point of view, we need to combine the forces of gravity and air resistance in Newton's second law. From a mathematical

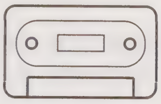
point of view, the main task is to solve first-order differential equations by the method of separation of variables.

In the next tape section we demonstrate how to solve problems which involve both gravity and air resistance. You should read through the following example before listening to the audio-tape.

For compatibility with references made on the audio-tape, the number sequences of the example and equations in this section follow on from those of Section 3.

Example 3

A particle of mass m is dropped from rest under the actions of gravity and an air resistance force which is assumed to be proportional to the particle's speed. Find how the velocity of the particle and the distance it falls vary with time.



Tape Section 3

Start the audio-tape when you are ready.

Working Notes for Example 3

1. Forces acting on object?

2. Initial conditions?

3. What are we asked to find?

4. Strategy for solution?

Solution to Example 3

We choose an x-axis pointing vertically downwards with its origin at the point where the particle is released. The forces acting on the particle are

1. the force of gravity, of magnitude mg downwards;
2. air resistance, of magnitude $R = kv$ upwards (note that v is positive throughout the motion, so that the speed here is $|v| = v$).

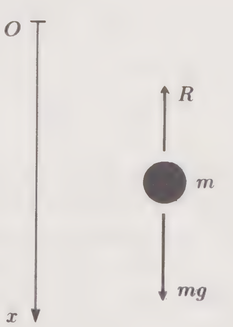


Figure 2

Newton's second law gives

$$m\frac{dv}{dt} = mg - kv.$$

We can solve this differential equation by the separation of variables method, obtaining

$$\int \frac{m}{mg - kv}\,dv = \int 1\,dt + c_1$$

or

$$-\frac{m}{k}\log_e(mg - kv) = t + c_1. \qquad (5)$$

$$\int \frac{1}{z}\,dz = \begin{cases} \log_e z & z > 0, \\ \log_e(-z) & z < 0. \end{cases}$$

In performing the integration on the left-hand side we have assumed that $mg - kv > 0$ throughout the motion (this is equivalent to $v < mg/k$). Initially (when $t = 0$) we have $v = 0$, so the value of the constant c_1 is given by

$$c_1 = -\frac{m}{k}\log_e(mg).$$

Hence

$$t = \frac{m}{k}\left(\log_e(mg) - \log_e(mg - kv)\right)$$
$$= \frac{m}{k}\log_e\left(\frac{mg}{mg - kv}\right).$$

We now seek to make v the subject of the formula. Multiplying through by k/m and then taking the exponential of each side produces

$$\frac{mg}{mg - kv} = e^{kt/m}$$

or, after some rearrangement,

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right). \tag{6}$$

We shall consider this equation further shortly, but note that it predicts $v < mg/k$ for all t, which is consistent with the assumption made in obtaining Equation (5). Writing dx/dt for v gives

$$\frac{dx}{dt} = \frac{mg}{k}\left(1 - e^{-kt/m}\right),$$

from which we obtain

$$x = \frac{mg}{k}t + \frac{m^2 g}{k^2}e^{-kt/m} + c_2.$$

The initial condition $x = 0$ when $t = 0$ leads to $c_2 = -m^2 g/k^2$, so that

$$x = \frac{mg}{k}t - \frac{m^2 g}{k^2}\left(1 - e^{-kt/m}\right). \tag{7}$$

This concludes the solution to Example 3. □

According to this model, in which air resistance was assumed proportional to the speed, the velocity is predicted to vary with time as described by Equation (6), namely

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right). \tag{8}$$

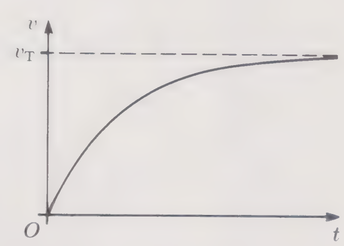


Figure 3
The graph of velocity v against time t for an object falling under the influence of gravity and linear air resistance.

The graph of this velocity function is shown in Figure 3. The speed increases but steadily approaches the limiting value $v_T = mg/k$. This value is called the *terminal speed* of the object and can be obtained by taking the limit of Equation (8) for large times t. Alternatively it can be derived directly from Newton's first law, which states that when a particle is moving with constant velocity then the total force acting on the particle is zero. In our case this implies that

$$mg - kv_T = 0$$

(see Figure 4) or that

$$v_T = \frac{mg}{k}.$$

Figure 4

Although theoretically a particle falling from rest never achieves its terminal speed, it reaches 98.2% of this speed after a time $t = 4m/k$ and 99.3% after a time $t = 5m/k$. In the television programme the ball-bearing dropped into a tank of glycerine had mass $m = 1.05 \times 10^{-3}$ kg and diameter $d = 6.30 \times 10^{-3}$ metres. For a sphere of this size falling through glycerine, the linear resistive force constant is $k = 8.88 \times 10^{-2}\,\mathrm{kg\,s^{-1}}$. The terminal speed is therefore predicted to be

$$v_T = \frac{mg}{k} = \frac{1.05 \times 10^{-3} \times 9.81}{8.88 \times 10^{-2}} \simeq 0.116\,\mathrm{m\,s^{-1}},$$

which agrees well with the experimental value.

Exercise 1

A parachutist of mass m opens his parachute when his speed is $2mg/k$, where k is a constant. Assuming that the air resistance on the parachutist is then proportional to his speed, with constant of proportionality k, and that the speed is always greater than mg/k, find an expression for his speed in terms of the time after the parachute was opened.

Exercise 2

Consider the motion of a particle of mass m which is dropped from rest under the action of gravity and a quadratic air resistance force of magnitude kv^2, where v is the downward velocity of the particle and k is a constant.

(i) Use Newton's first law to find the terminal speed v_{T} of the particle.

(ii) Use Newton's second law to show that if the particle is dropped (from rest) at time $t = 0$ and the x-axis is chosen to point downwards then the velocity function of the particle is

$$v(t) = v_{\mathrm{T}} \left(\frac{e^{2t/\tau} - 1}{e^{2t/\tau} + 1} \right),$$

where $\tau = v_{\mathrm{T}}/g$. (You may assume that $v < v_{\mathrm{T}}$ throughout the motion. The first integral given in the Hint below will be of use here, and it will help to express mg in terms of v_{T} at an early stage.)

(iii) Use Newton's second law to show that if the particle is dropped from the origin $x = 0$ then

$$v(x) = v_{\mathrm{T}} \left(1 - e^{-2kx/m}\right)^{1/2}.$$

[Hint: $\displaystyle\int \frac{1}{a^2 - z^2}\, dz = \frac{1}{2a} \log_e \left(\frac{a+z}{a-z} \right)$ for $|z| < |a|$;

$\displaystyle\int \frac{z}{a^2 - z^2}\, dz = -\frac{1}{2} \log_e (a^2 - z^2)$ for $|z| < |a|$.]

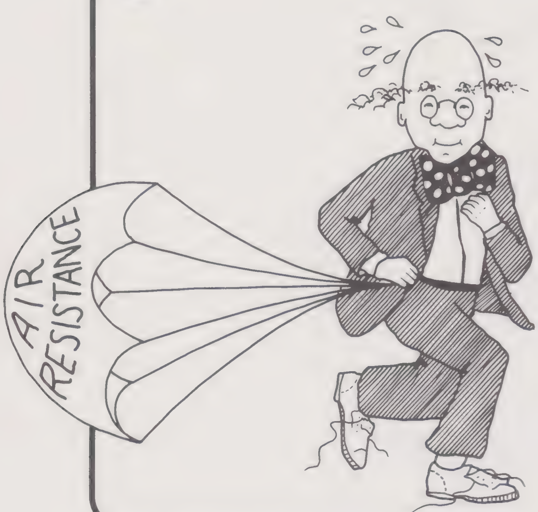
[*Solutions on page 54*]

The example and exercises so far in this subsection have concerned situations involving air resistance in which the motion is in the direction of increasing x and so has positive velocity. In cases where the velocity may be negative some care is required when applying Newton's second law to write down the equation of motion. The following audio-tape section deals with the various possibilities that can arise.

Start the audio-tape when you are ready.

Tape Section 4

1 Air Resistance



Direction of force is

→ opposite ←

to direction of motion

magnitude R

(i) Linear Approximation

$$\boxed{R = k|v|}$$

(low speeds)

(ii) Quadratic Approximation

$$\boxed{R = k|v|^2}$$

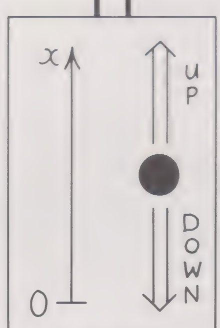
(high speeds)

2 Linear Resistance

$$R = k|v|$$

$$|v| = \begin{cases} \square\, v & (v \geq 0) \\ \square\, v & (v < 0) \end{cases}$$

$$R = \begin{cases} \square\, kv & (v \geq 0) \\ \square\, kv & (v < 0) \end{cases}$$



3 Quadratic Resistance

$$R = k|v|^2$$

$$|v|^2 = \begin{cases} \square\, v^2 & (v \geq 0) \\ \square\, v^2 & (v < 0) \end{cases}$$

$$R = \begin{cases} \square\, kv^2 & (v \geq 0) \\ \square\, kv^2 & (v < 0) \end{cases}$$

4 Equation of Motion

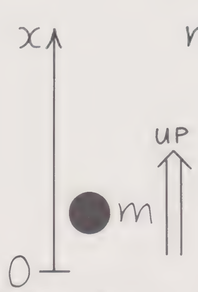
magnitude and direction

Forces: (a) Gravity → mg downwards

(b) Air resistance R opposite to direction of motion

GOING UP OR DOWN ?

(i) Upward Motion

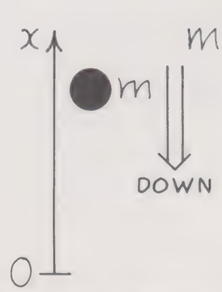


$$m\frac{dv}{dt} = F$$

$$= \boxed{}$$

v is $\square$ +ve or −ve ?

(ii) Downward Motion



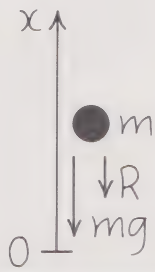
$$m\frac{dv}{dt} = F$$

$$= \boxed{}$$

v is $\square$ +ve or −ve ?

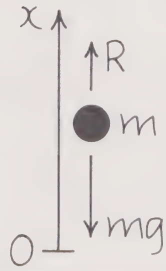
5 Linear Resistance: $R = k|v|$

(i) Upward Motion (v is +ve)



$$m\frac{dv}{dt} = -mg - R$$
$$= -mg - k|v|$$
$$m\frac{dv}{dt} = \boxed{}$$

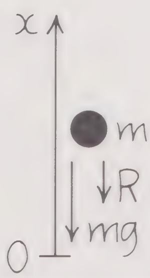
(ii) Downward Motion (v is −ve)



$$m\frac{dv}{dt} = -mg + R$$
$$= -mg + k|v|$$
$$m\frac{dv}{dt} = \boxed{}$$

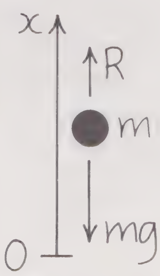
6 Quadratic Resistance: $R = k|v|^2$

(i) Upward Motion (v is +ve)



$$m\frac{dv}{dt} = -mg - R$$
$$= -mg - k|v|^2$$
$$m\frac{dv}{dt} = \boxed{}$$

(ii) Downward Motion (v is −ve)



$$m\frac{dv}{dt} = -mg + R$$
$$= -mg + k|v|^2$$
$$m\frac{dv}{dt} = \boxed{}$$

In the above tape section you saw that, for a particle moving vertically subject to quadratic air resistance, the equation of motion depended on whether the motion was upward or downward. For linear air resistance on the other hand, the form of the equation of motion was unaffected by the direction of travel. You will see situations analogous to the second case for other linear forces in *Units 7* and *8*.

The single equation for both upward and downward motion in the linear case can be explained in terms of force x-components as follows. Suppose, as before, that the x-axis is directed vertically upwards. Then Newton's second law (for upward or downward motion) is

$$m\frac{dv}{dt} = -mg + A, \tag{9}$$

where $-mg$ is the x-component of the gravitational force, which always acts in the negative x-direction, and A is the x-component of the air resistance force. Now the linear air resistance force has magnitude $R = k|v|$ and direction opposite to that of v (or equivalently, direction coinciding with that of $-v$). Consequently we have

$$A = -kv$$

whether v is positive or negative, giving the single differential equation

$$m\frac{dv}{dt} = -mg - kv.$$

For the quadratic case, the motion is again described by Equation (9). Here the air resistance force has magnitude $R = k|v|^2 = kv^2$ and direction opposite to that of v, so its x-component is

$$A = \begin{cases} -kv^2 & v \geq 0, \\ kv^2 & v < 0, \end{cases}$$

which leads to two distinct differential equations.

Exercise 3

A particle moves vertically under the forces of gravity and quadratic air resistance. The x-axis is chosen to point downwards. Use Newton's second law to find the equation of motion when the particle is moving

(i) downwards;

(ii) upwards.

[*Solution on page 55*]

Two models for a falling object's motion have been developed in this and the previous section. The first assumed that gravity was the only force acting, and the second also took air resistance into account. However, even the revised model does not tell the whole story. Objects which are sufficiently light will rise when released in a fluid rather than fall, due to pressure variation within the fluid. Thus a hollow rubber ball immersed in water will rapidly rise to the surface when released. The same effect is put to use for the purpose of flight in a hot-air balloon. The motion of such an object may be modelled more satisfactorily by allowing for the presence of a third force in addition to gravity and air (or other fluid) resistance. This is known as *Archimedes' buoyancy force*, and its magnitude is equal to the weight of fluid displaced by the object. The effect of this force is not significant for the situations considered earlier, and may safely be ignored for objects which are relatively heavy.

Further effects due to the motion of the fluid may also need to be considered in certain circumstances, but fluid dynamics is beyond the scope of this course.

Summary of Section 4

In this section Newton's second law has been used to model the motion of an object falling under the forces of gravity and air resistance.

1. The motion of any object through the Earth's atmosphere is opposed by the force of **air resistance**, whose magnitude R depends on the object's speed $|v|$, shape and size. For small speeds it is appropriate to use the linear approximation

$$R \simeq k_1|v|,$$

whereas for larger speeds the quadratic approximation

$$R \simeq k_2|v|^2$$

is appropriate. Here k_1 and k_2 are constants which depend on the object's shape and size.

2. The constant speed $|v_T|$ at which an object can fall is called the **terminal speed**. Using Newton's first law, this occurs when $R = mg$.

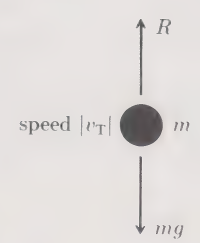


Figure 5

5 End of unit exercises

Section 1

Exercise 1

A particle moves along an x-axis so that its position x at time t is given by

$$x(t) = t^4 - 3t^2 + 1 \quad (t \geq 0).$$

(i) Find the particle's position, velocity, speed and acceleration at the instant $t = 1$.

(ii) What is the direction of motion at the instant $t = 1$? Is the particle speeding up or slowing down at this time?

Exercise 2

At time $t = 0$, two particles are set moving along the same straight line. Particle A has acceleration $a_A(t) = t + 2$, and initially its position and velocity are $x_A(0) = 0$ and $v_A(0) = 1$. Particle B has acceleration $a_B(t) = t - 1$, and initially its position and velocity are $x_B(0) = 8$ and $v_B(0) = 0$.

When and where do these particles collide after their release?

Exercise 3

A car, which is initially at rest at the origin, accelerates uniformly for t_0 seconds until its velocity is v_0. It then continues to travel with constant velocity v_0. Find expressions for its position x and velocity v at time t in terms of the constants t_0 and v_0.

[*Solutions on page 56*]

Section 2

Exercise 4

A lift weighing 1000 kg starts upwards with constant acceleration, and attains a velocity of $3\,\text{m}\,\text{s}^{-1}$ after 2 seconds. Find the tension in the supporting cable during this accelerated motion, ignoring any effects due to air resistance or friction between the lift and the sides of the lift shaft.

Exercise 5

The position function of a particle of unit mass (that is, $m = 1$) is

$$x(t) = e^{-3t} \sin 4t.$$

Show that the force acting on the particle has x-component

$$F = -25x - 6v,$$

where v is the particle's velocity.

Exercise 6

An object of mass m is placed on a horizontal plane. Find the magnitude R of the reaction force exerted on the object by the plane when the plane has

(i) an *upward* vertical acceleration of magnitude a;

(ii) a *downward* vertical acceleration of magnitude a (assuming $a < g$).

[*Solutions on page 57*]

Section 3

Exercise 7

A particle of mass m moves under the action of a force with x-component

$$F = -m\omega^2 x,$$

where ω is a constant. If the particle is at the origin $x = 0$ with velocity $v = v_0$ (where $v_0 > 0$) at time $t = 0$, find how the velocity of the particle depends on position x. Hence show that the position function of the particle is

$$x = \frac{v_0}{\omega} \sin \omega t.$$

Exercise 8

A particle of mass m moves under the action of a force with x-component

$$F = kxv^2,$$

where k is a constant. If the particle is initially at the origin $x = 0$ with velocity $v = v_0$ (where $v_0 > 0$), show that the velocity v at position x is given by

$$v = v_0 \exp\left(\frac{kx^2}{2m}\right).$$

Exercise 9

A particle is projected vertically upwards with initial speed v_0 at the instant $t = 0$. At the instant $t = T$ (where $0 < T < 2v_0/g$) a second particle is projected vertically upwards from the same point and with the same initial speed. Show that, if air resistance is negligible, the two particles will collide at the instant

$$t = \frac{T}{2} + \frac{v_0}{g}.$$

[*Solutions on page 57*]

Section 4

Exercise 10

A particle of mass m is projected vertically upwards with initial speed v_0. It moves under the action of gravity and quadratic air resistance of magnitude mkv^2, where k is a constant and v is the velocity of the particle. When the particle returns to the point of projection its speed is v_1.

(i) Show that during its upward motion the particle's velocity v at height x above the point of projection is given by

$$v^2 = \left(\frac{g}{k} + v_0^2\right) e^{-2kx} - \frac{g}{k}.$$

(ii) Show that during the particle's downward motion its velocity v at height x is given by

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} - v_1^2\right) e^{2kx}.$$

(iii) Hence show that

$$\frac{1}{v_1^2} = \frac{1}{v_0^2} + \frac{k}{g}.$$

[*Solution on page 58*]

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. The positions of particles A, B and C are $x_A = -2$, $x_B = 1$ and $x_C = 2.5$ respectively.

The distances between the particles are

$$x_B - x_A = 1 - (-2) = 3 \text{ metres}$$

between A and B,

$$x_C - x_A = 2.5 - (-2) = 4.5 \text{ metres}$$

between A and C, and

$$x_C - x_B = 2.5 - 1 = 1.5 \text{ metres}$$

between B and C.

2. **(i)**

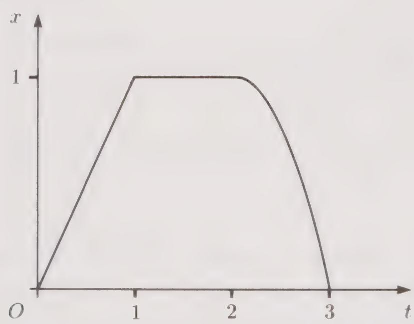


Figure 1

(ii) Initially, at $t = 0$, the particle is at the origin $x = 0$. It moves in the direction of increasing x until it reaches position $x = 1$ at $t = 1$. It remains at this point until $t = 2$. It then moves in the direction of decreasing x until it finally returns to the origin at $t = 3$.

3. **(i)** The average velocity during the time interval from $t = 2$ to $t = 3$ is (in metres per second)

$$\frac{x(3) - x(2)}{3 - 2} = \frac{12 - 6}{1} = 6.$$

(ii) The average velocity during the time interval from $t = 2$ to $t = 2.1$ is (in metres per second)

$$\frac{x(2.1) - x(2)}{2.1 - 2} = \frac{6.51 - 6}{0.1} = 5.1.$$

(iii) The average velocity during the time interval from $t = 2$ to $t = 2.01$ is (in metres per second)

$$\frac{x(2.01) - x(2)}{2.01 - 2} = \frac{6.0501 - 6}{0.01} = 5.01.$$

(iv) The average velocity during the time interval from $t = 2$ to $t = 2.001$ is (in metres per second)

$$\frac{x(2.001) - x(2)}{2.001 - 2} = \frac{6.005\,001 - 6}{0.001} = 5.001.$$

4. The velocity function is

$$v(t) = \frac{dx}{dt} = 2t - 4.$$

(i) $v(4) = 2 \times 4 - 4 = 4$,

so the particle has velocity $4\,\text{m}\,\text{s}^{-1}$ after 4 seconds.

(ii) $v(1) = 2 \times 1 - 4 = -2$,

so the particle has velocity $-2\,\text{m}\,\text{s}^{-1}$ after 1 second.

(iii) The particle is instantaneously at rest when $v(t) = 0$. This occurs when $2t - 4 = 0$, or $t = 2$. So the particle is momentarily at rest after 2 seconds.

5. The velocity function is

$$v(t) = \frac{dx}{dt} = 3t^2 - 12.$$

(i) $v(1) = -9$, $v(2) = 0$, $v(3) = 15$, so the velocities of the particle at times $t = 1$, $t = 2$ and $t = 3$ are $-9\,\text{m}\,\text{s}^{-1}$, $0\,\text{m}\,\text{s}^{-1}$ and $15\,\text{m}\,\text{s}^{-1}$ respectively, whereas the speeds are $9\,\text{m}\,\text{s}^{-1}$, $0\,\text{m}\,\text{s}^{-1}$ and $15\,\text{m}\,\text{s}^{-1}$ respectively.

(ii)

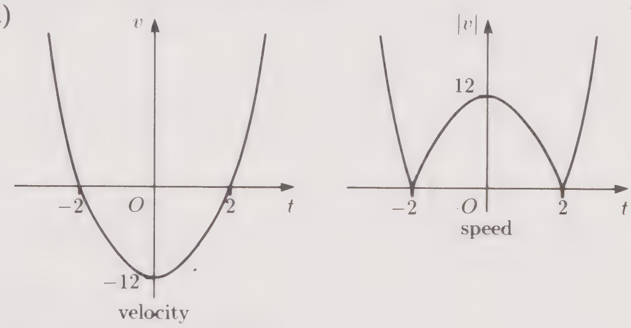


Figure 2

6. **(i)** The position function is $x(t) = te^{-t}$. Using the chain rule for differentiation gives the velocity function

$$v(t) = \frac{dx}{dt} = (1)(e^{-t}) + (t)(-e^{-t})$$
$$= (1 - t)e^{-t},$$

and differentiating once more leads to the acceleration function

$$a(t) = \frac{dv}{dt} = (-1)(e^{-t}) + (1 - t)(-e^{-t})$$
$$= (t - 2)e^{-t}.$$

(ii)

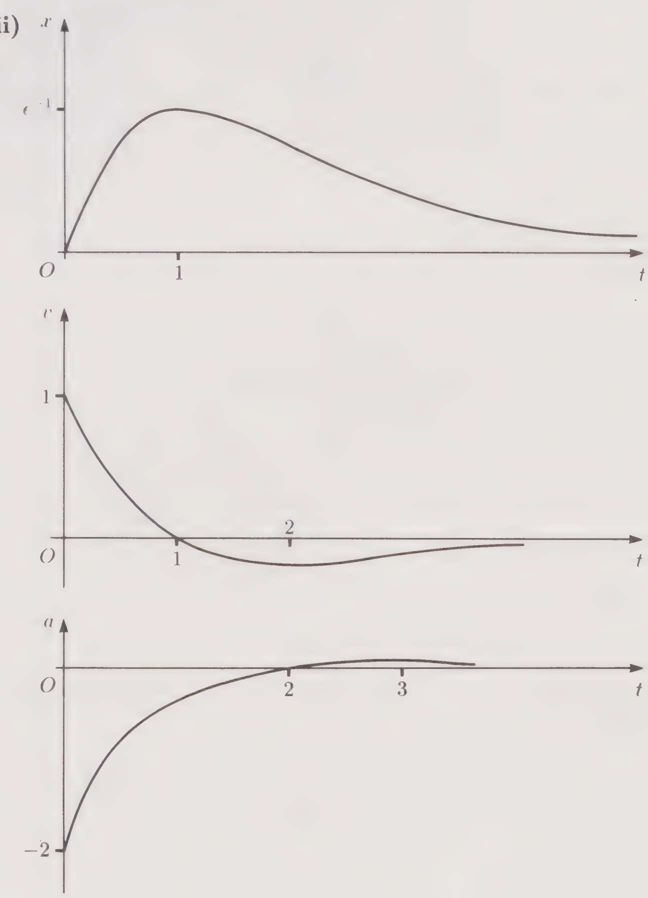


Figure 3

(iii) From the second and third sketch graphs it can be seen that the acceleration is positive for $t > 2$, while the speed is decreasing for the whole of this interval.

7. Since

$$a(t) = \frac{dv}{dt} = 18t - 20,$$

we have

$$v(t) = \int (18t - 20)\,dt + c_1$$
$$= 9t^2 - 20t + c_1.$$

Using the initial condition $v(0) = 3$, we obtain $c_1 = 3$. Hence the velocity function of the particle is

$$v(t) = 9t^2 - 20t + 3.$$

Now $dx/dt = v(t)$, so that

$$x(t) = \int (9t^2 - 20t + 3)\,dt + c_2$$
$$= 3t^3 - 10t^2 + 3t + c_2.$$

The initial condition $x(0) = 7$ leads to $c_2 = 7$. Hence the position function is

$$x(t) = 3t^3 - 10t^2 + 3t + 7.$$

Substituting $t = 10$ into this expression produces

$$x(10) = 3000 - 1000 + 30 + 7 = 2037,$$

so the position of the particle is 2037 m at time $t = 10$.

8. On integrating the acceleration function

$$a(t) = \frac{dv}{dt} = e^{-kt},$$

we obtain

$$v(t) = \int e^{-kt}dt + c_1$$
$$= c_1 - \frac{1}{k}e^{-kt}.$$

The initial condition $v(0) = 0$ gives $c_1 = 1/k$, so the velocity function of the particle is

$$v(t) = \frac{1}{k} - \frac{1}{k}e^{-kt}.$$

Then from $dx/dt = v(t)$ we have

$$x(t) = \int \left(\frac{1}{k} - \frac{1}{k}e^{-kt}\right) dt + c_2$$
$$= \frac{1}{k}t + \frac{1}{k^2}e^{-kt} + c_2.$$

The initial condition $x(0) = 0$ leads to $c_2 = -1/k^2$. So finally we have the position function

$$x(t) = \frac{1}{k}t + \frac{1}{k^2}e^{-kt} - \frac{1}{k^2}.$$

9. (i) The acceleration function is

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(v_0) = 0,$$

since v_0 is a constant.

(ii) From the given velocity function

$$v(t) = \frac{dx}{dt} = v_0,$$

integration produces

$$x(t) = \int v_0\,dt + c = v_0t + c.$$

The initial condition $x(0) = x_0$ gives $c = x_0$, so the position function is

$$x(t) = x_0 + v_0t.$$

(iii) The sketch graphs are shown in Figure 10 of the main text, which is in the margin following Exercise 9.

10. (i) The acceleration function is

$$a(t) = \frac{dv}{dt} = a_0,$$

which on integration yields

$$v(t) = \int a_0\,dt + c_1 = a_0t + c_1.$$

The initial condition $v(0) = v_0$ gives $c_1 = v_0$, so the velocity function of the particle is

$$v(t) = v_0 + a_0t. \qquad (1)$$

Hence

$$\frac{dx}{dt} = v_0 + a_0t,$$

from which

$$x(t) = \int (v_0 + a_0t)\,dt + c_2$$
$$= v_0t + \tfrac{1}{2}a_0t^2 + c_2.$$

Using the initial condition $x(0) = x_0$ we obtain $c_2 = x_0$, and so the position function is

$$x(t) = x_0 + v_0t + \tfrac{1}{2}a_0t^2. \qquad (2)$$

(ii) From Equation (1) we have, on squaring both sides,

$$v^2 = (v_0 + a_0t)^2$$
$$= v_0^2 + 2v_0a_0t + a_0^2t^2. \qquad (3)$$

On multiplying through Equation (2) by $2a_0$ we have

$$2a_0x = 2a_0x_0 + 2a_0v_0t + a_0^2t^2. \qquad (4)$$

Subtraction of Equation (4) from Equation (3) followed by some rearrangement gives

$$v^2 = v_0^2 + 2a_0(x - x_0),$$

which is the required result.

11. Using the given expression for the acceleration, we have

$$v\frac{dv}{dx} = a_0.$$

This differential equation can be solved by the separation of variables method, giving

$$\int v\,dv = \int a_0\,dx + c$$

or $\quad \tfrac{1}{2}v^2 = a_0x + c.$

The initial condition $v = v_0$ at $x = x_0$ gives

$$c = \tfrac{1}{2}v_0^2 - a_0x_0.$$

Hence

$$\tfrac{1}{2}v^2 = a_0x + (\tfrac{1}{2}v_0^2 - a_0x_0).$$

Rearranging this equation leads to the required result

$$v^2 = v_0^2 + 2a_0(x - x_0).$$

Solutions to the exercises in Section 2

1. (i) Any moving car is subject to resistive forces, namely air resistance and internal frictional forces in the car's engine, transmission and wheel bearings. In order to maintain a constant velocity it is necessary to apply a motive force which exactly balances these resistive forces. Thus a car moving at constant velocity has no net force acting upon it.

(ii) The toboggan is on a slope, and the gravitational force of attraction due to the Earth has a component down the slope. If the slope is steep enough then this force down the slope will be greater than the resistive forces of friction and air resistance, causing the toboggan to accelerate.

2. The acceleration is

$$a = \frac{dv}{dt} = g.$$

Hence

$$v = gt + c_1.$$

Initially, $v = 0$ when $t = 0$, which results in the constant of integration c_1 being zero. So we have

$$v = \frac{dx}{dt} = gt.$$

Integrating this gives

$$x = \tfrac{1}{2}gt^2 + c_2.$$

The initial condition $x = 0$ when $t = 0$ gives $c_2 = 0$. Hence

$$x = \tfrac{1}{2}gt^2.$$

3. Differentiation of

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

produces

$$\frac{dv}{dt} = ge^{-kt/m}.$$

Also

$$\begin{aligned} mg - kv &= mg - mg(1 - e^{-kt/m}) \\ &= mge^{-kt/m}. \end{aligned}$$

Hence

$$m\frac{dv}{dt} = mge^{-kt/m} = mg - kv.$$

Also at $t = 0$ we have

$$v = \frac{mg}{k}(1 - e^0) = 0.$$

Hence

$$v = \frac{mg}{k}(1 - e^{-kt/m})$$

is the required solution of the differential equation with the given initial condition.

4. If $F = ma$ then zero acceleration implies zero force. In view of the efforts made to reduce the effects of friction, one expects any force along the track to be very small. So it is not surprising that the glider moves with no discernible acceleration, like a particle obeying Newton's first law.

5. **(i)** From the after-images in Figure 1 of Section 2, the following data about the positions of the glider were obtained.

t s	0	0.75	1.5	2.25	3.0
x m	0	0.12	0.43	0.96	1.72

In Figure 1(a) these points have been plotted and a smooth curve drawn between them.

(ii) Tangents were drawn to obtain values of the instantaneous velocity every $\frac{3}{4}$ second. To illustrate this method, one of these tangents has been drawn in Figure 1(a). The following data were obtained for the velocity of the glider.

t s	0	0.75	1.5	2.25	3.0
v m s^{-1}	0	0.29	0.575	0.84	1.14

In Figure 1(b), these points have been plotted by a straight line.

(iii) The straight line in Figure 1(b) has a slope of $0.38\,\mathrm{m\,s^{-2}}$. In view of the inaccuracies in releasing the glider from rest and plotting the data, this *is* consistent with the claim made in the programme that $a = 0.4$ (in $\mathrm{m\,s^{-2}}$).

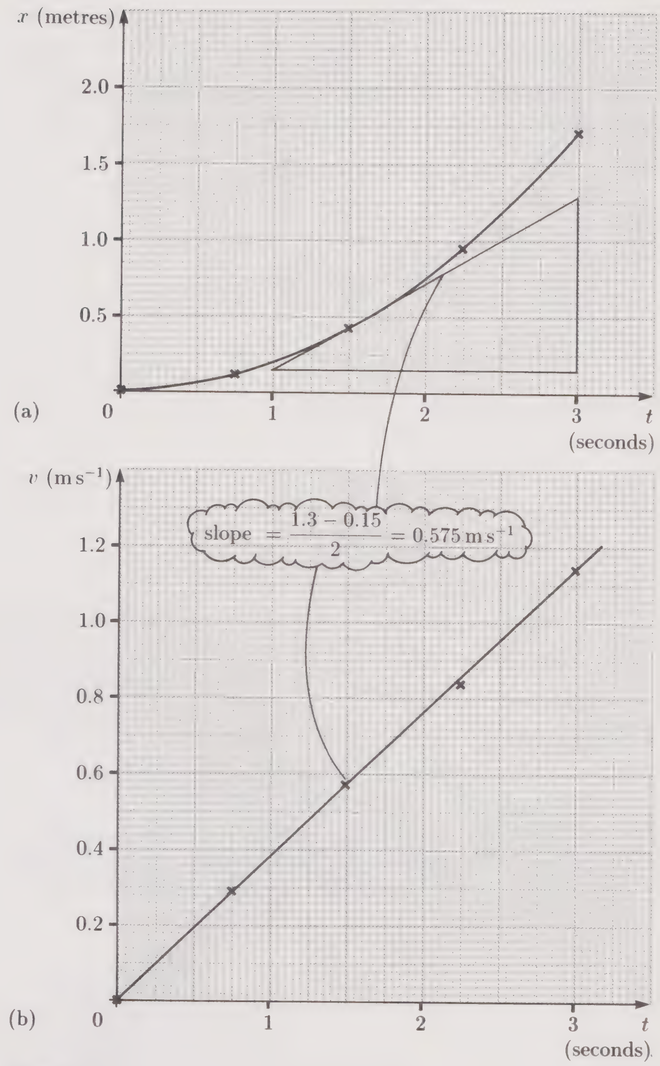


Figure 1

6. For the second glider,

$$F = 1 \times 0.5 = 0.5 \text{ force units.}$$

For the first glider,

$$F = m \times 0.4 \text{ force units.}$$

Hence

$$m = 0.5/0.4 = 1.25 \text{ mass units.}$$

7. A net force of 0.11 force units up the track should cause an object of mass 1.5 units to have an acceleration of $0.11/1.5 \simeq 0.07\,\mathrm{m\,s^{-2}}$ up the track. This agrees with the acceleration observed in the programme.

8. In cases (i), (ii) and (iii) the glider has zero acceleration, and so it experiences no net force. The additional force of magnitude F up the slope must therefore cancel the force acting down the slope. Thus $F = 0.09$ force units.

In case (iv) there is a net force of magnitude $0.09 - F$ acting down the slope. This causes an acceleration of $0.02\,\mathrm{m\,s^{-2}}$ for a glider of mass 1.5 units. Hence

$$0.09 - F = 1.5 \times 0.02 = 0.03,$$

and $F = 0.09 - 0.03 = 0.06$ force units.

9. As the object is at rest, and so has zero acceleration, the net force acting on it must have magnitude zero. Hence the magnitude of the downward gravitational force is equal to the magnitude of the upward spring force.

10. When the car is moving with constant speed, the motive force exactly balances the resistive forces. But these resistive forces, particularly air resistance, increase with speed. So we must increase the motive force in order to maintain a higher constant speed.

11. **(i)** $F = 2\,\mathrm{N}$

(ii) $F = -2\,\mathrm{N}$

(iii) $F = 2 + 5 = 7\,\mathrm{N}$

(iv) $F = 5 - 2 = 3\,\mathrm{N}$

(v) $F = 2 + 3 - 6 = -1\,\mathrm{N}$

12. **(i)**

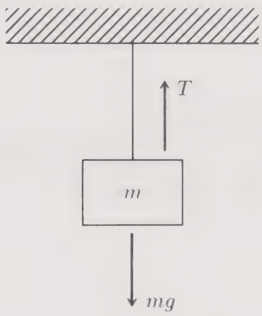


Figure 2

There is a downward force on the object of magnitude mg, namely the gravitational force. So by Newton's first law there must also be an upward force of the same magnitude. This force can be caused only by the string; its magnitude is called the *tension* in the string. If we denote the tension by T then we have

$$T = mg.$$

(ii)

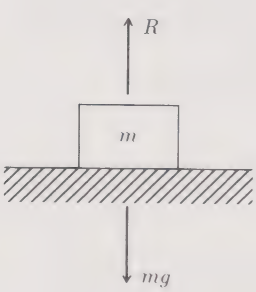


Figure 3

There is a downward force on the object of magnitude mg, namely the gravitational force. So there must be a second force of the same magnitude acting upwards. This upward thrust, caused by the surface contact with the table, is usually called the *reaction* force. If we denote its magnitude by R then we have

$$R = mg.$$

Solutions to the exercises in Section 3

1. In each case the force F is a function of time t only, which suggests that we should use the expression $a = dv/dt$ for the acceleration.

(i) Newton's second law $F = ma$ leads to

$$\frac{dv}{dt} = kt^2.$$

Integrating this gives

$$v = \tfrac{1}{3}kt^3 + c_1,$$

where c_1 is a constant of integration. Using the initial condition $v = 0$ when $t = 0$ gives $c_1 = 0$. Hence the velocity function is

$$v = \tfrac{1}{3}kt^3.$$

In order to find x as a function of time, we use $v = dx/dt$. This gives

$$\frac{dx}{dt} = \tfrac{1}{3}kt^3.$$

Integrating, we obtain

$$x = \tfrac{1}{12}kt^4 + c_2,$$

where c_2 is a constant of integration. Initially, $x = 0$ when $t = 0$, which leads to $c_2 = 0$. Hence the position function is

$$x = \tfrac{1}{12}kt^4.$$

(ii) Newton's second law $F = ma$ gives

$$\frac{dv}{dt} = \frac{k}{m}\sin\omega t.$$

Hence

$$v = c_1 - \frac{k}{m\omega}\cos\omega t.$$

From the initial condition $v = 0$ when $t = 0$ we have

$$c_1 = \frac{k}{m\omega},$$

so the velocity function is

$$v = \frac{k}{m\omega} - \frac{k}{m\omega}\cos\omega t.$$

Hence

$$\frac{dx}{dt} = \frac{k}{m\omega} - \frac{k}{m\omega}\cos\omega t.$$

Integrating, we obtain

$$x = \frac{k}{m\omega}t - \frac{k}{m\omega^2}\sin\omega t + c_2.$$

The initial condition $x = 0$ when $t = 0$ leads to $c_2 = 0$. So the position function is

$$x = \frac{k}{m\omega}t - \frac{k}{m\omega^2}\sin\omega t.$$

2. In each case the force F is a function of position x only, which suggests that we use the expression $a = v\,dv/dx$ for the acceleration.

(i) Newton's second law $F = ma$ gives

$$v\frac{dv}{dx} = \frac{k}{m}\cos\omega x.$$

This differential equation is soluble by the separation of variables method, leading to

$$\int v\,dv = \int \frac{k}{m}\cos\omega x\,dx + c,$$

that is,

$$\tfrac{1}{2}v^2 = \frac{k}{m\omega}\sin\omega x + c.$$

The particle starts from rest at the origin, so initially $v = 0$ when $x = 0$. This gives $c = 0$. Hence

$$v = \pm\left(\frac{2k}{m\omega}\sin\omega x\right)^{1/2}.$$

The $+$ and $-$ signs here apply alternately as the particle oscillates between $x = 0$ and $x = \pi/\omega$.

(ii) Newton's second law $F = ma$ leads to

$$v\frac{dv}{dx} = \frac{k}{m}e^{bx}.$$

Separating and integrating this differential equation gives

$$\int v\,dv = \int \frac{k}{m}e^{bx}\,dx + c$$

or $\quad \tfrac{1}{2}v^2 = \dfrac{k}{mb}e^{bx} + c.$

The initial condition $v = 0$ when $x = 0$ gives $c = -k/mb$, and so

$$v = \left(\frac{2k}{mb}(e^{bx} - 1)\right)^{1/2}.$$

(After the motion starts, the velocity is always positive, since the acceleration is positive and $v = 0$ at the outset.)

3. **(i)** As we wish to obtain v as a function of t, the appropriate expression for the acceleration is $a = dv/dt$. Newton's second law of motion then gives

$$\frac{dv}{dt} = -\frac{k}{m}v^2.$$

This differential equation is soluble by the separation of variables method, leading to

$$-\int \frac{1}{v^2}\,dv = \frac{k}{m}\int 1\,dt + c_1$$

or $\displaystyle \frac{1}{v} = \frac{k}{m}t + c_1.$

Initially, $v = v_0$ when $t = 0$, and so $c_1 = 1/v_0$. Hence

$$\frac{1}{v} = \frac{k}{m}t + \frac{1}{v_0}.$$

Rearranging this equation, we obtain the velocity function

$$v = \frac{mv_0}{m + kv_0 t}. \qquad (1)$$

Hence

$$\frac{dx}{dt} = \frac{mv_0}{m + kv_0 t}.$$

On integrating we find that the position function is

$$x = \frac{m}{k}\log_e(m + kv_0 t) + c_2.$$

The initial condition $x = 0$ when $t = 0$ gives $c_2 = -(m/k)\log_e m$. Hence

$$x = \frac{m}{k}\log_e(m + kv_0 t) - \frac{m}{k}\log_e m$$
$$= \frac{m}{k}\log_e\left(\frac{m + kv_0 t}{m}\right). \qquad (2)$$

Note from Equation (1) that the velocity of the puck decreases to zero for large t, as might be expected, but according to Equation (2) the position x increases indefinitely.

(ii) As we wish to obtain v as a function of x, the appropriate expression for the acceleration is $a = v\,dv/dx$. Newton's second law then gives

$$v\frac{dv}{dx} = -\frac{k}{m}v^2.$$

Separating and integrating this differential equation gives

$$\int \frac{1}{v}\,dv = -\frac{k}{m}\int 1\,dx + c$$

or $\displaystyle \log_e v = -\frac{k}{m}x + c.$

Initially, $v = v_0$ when $x = 0$, giving $c = \log_e v_0$. Hence

$$\log_e v = -\frac{k}{m}x + \log_e v_0,$$

which on taking exponentials of both sides leads to

$$v = v_0 e^{-kx/m}.$$

(The same result could have been obtained by eliminating t between Equations (1) and (2).)

4. **(i)** Newton's second law is $F = ma$, or

$$a = \frac{1}{m}F(t, v).$$

As the right-hand side is a function of t and v, an appropriate expression for the acceleration is

$$a = \frac{dv}{dt}.$$

Integrating the resulting differential equation would give v as a function of t.

(ii) Newton's second law is

$$a = \frac{1}{m}F(x, v).$$

The right-hand side is a function of x and v, which suggests that we use the expression

$$a = v\frac{dv}{dx}.$$

Solving the resulting differential equation would give v as a function of x.

5.

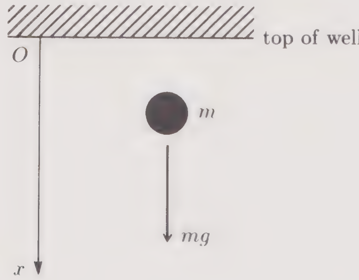


Figure 1

We assume that the only force acting on the stone is gravity, of magnitude mg downwards. The x-axis is chosen to point vertically downwards with the origin at the top of the well. (This is an arbitrary choice which can have no effect on the final answers to the question.) Applying Newton's second law,

$$ma = F = mg, \quad \text{or} \quad a = g.$$

Hence

$$\frac{dv}{dt} = g.$$

Integrating this differential equation, we obtain

$$v = gt + c_1.$$

The initial condition $v = 0$ (stone dropped from rest) when $t = 0$ leads to $c_1 = 0$. Hence

$$v = gt. \qquad (3)$$

Therefore

$$\frac{dx}{dt} = gt.$$

Integrating, we obtain

$$x = \tfrac{1}{2}gt^2 + c_2.$$

The initial condition $x = 0$ when $t = 0$ gives $c_2 = 0$. So

$$x = \tfrac{1}{2}gt^2. \qquad (4)$$

(i) Using Equation (4), at $t = 3.00$ we have

$$x = \tfrac{1}{2} \times 9.81 \times 3.00^2 = 44.145.$$

(ii) Using Equation (3), at $t = 3.00$ we have

$$v = 9.81 \times 3.00 = 29.43.$$

Thus the depth of the well is estimated to be 44.1 m and the speed of the stone as it strikes the bottom is $29.4\,\mathrm{m\,s^{-1}}$.

6.

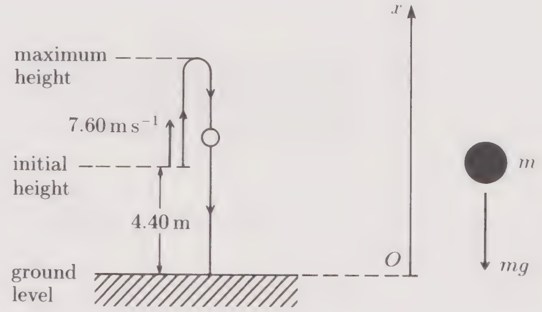


Figure 2

We assume that the only force acting on the ball is gravity, of magnitude mg downwards. The x-axis is chosen to point vertically upwards with the origin at ground level. Using Newton's second law,

$$ma = F = -mg, \quad \text{or} \quad a = -g.$$

Hence

$$\frac{dv}{dt} = -g.$$

Integrating this, we obtain

$$v = -gt + c_1.$$

The initial condition $v = 7.60$ at $t = 0$ gives $c_1 = 7.60$. Hence

$$v = -gt + 7.60. \tag{5}$$

Therefore

$$\frac{dx}{dt} = -gt + 7.60.$$

Integrating,

$$x = -\tfrac{1}{2}gt^2 + 7.60t + c_2.$$

The initial condition is $x = 4.40$ at $t = 0$, which leads to $c_2 = 4.40$. Hence, putting $g = 9.81$, we have

$$x = -4.905t^2 + 7.60t + 4.40. \tag{6}$$

(i) The ball reaches the ground when $x = 0$. Substituting this value into Equation (6), the corresponding time t satisfies the quadratic equation

$$4.905t^2 - 7.60t - 4.40 = 0,$$

whose solution is

$$\begin{aligned} t &= \frac{7.60 \pm \sqrt{7.60^2 + 4 \times 4.905 \times 4.40}}{2 \times 4.905} \\ &= 1.999 \quad \text{or} \quad -0.4479. \end{aligned}$$

The negative time is before the ball is thrown and may therefore be ignored. So the ball lands 2.00 seconds after being thrown.

(ii) Substituting this time into Equation (5) gives

$$v = -9.81 \times 1.999 + 7.60 = -12.01,$$

so the ball lands with a speed of $12.0\,\mathrm{m\,s^{-1}}$. (The negative sign for v confirms that it is then moving downwards.)

7. **(i)** Exactly as in the solution to Example 1, we choose an x-axis with the origin at the point of release and pointing vertically downwards. The only force acting on the object is gravity, of magnitude mg downwards. So Newton's second law gives

$$a = g.$$

Writing the acceleration a as $v\,dv/dx$, we have

$$v\frac{dv}{dx} = g.$$

This differential equation is soluble by the method of separation of variables. Hence

$$\int v\,dv = \int g\,dx + c_1$$

or $\quad \tfrac{1}{2}v^2 = gx + c_1.$

Now initially, $v = 0$ at $x = 0$, which leads to $c_1 = 0$. Hence

$$\tfrac{1}{2}v^2 = gx,$$

or $\quad v = \sqrt{2gx}, \qquad (7)$

where we have taken the positive square root because the velocity is positive throughout the motion.

(ii) Putting $v = dx/dt$ in Equation (7) gives

$$\frac{dx}{dt} = \sqrt{2gx}.$$

Again we use the method of separation of variables to solve this differential equation. So

$$\begin{aligned} \int 1\,dt &= \int \frac{1}{\sqrt{2gx}}\,dx + c_2 \\ &= \frac{1}{\sqrt{2g}} \int x^{-1/2}\,dx + c_2 \end{aligned}$$

or

$$t = \sqrt{\frac{2x}{g}} + c_2.$$

Initially, $x = 0$ at $t = 0$, and so $c_2 = 0$. Hence

$$t = \sqrt{\frac{2x}{g}}. \tag{8}$$

(iii) At $x = 77.0$, Equation (8) yields

$$t = \sqrt{\frac{2 \times 77.0}{9.81}} = 3.962$$

and Equation (7) gives

$$v = \sqrt{2 \times 9.81 \times 77.0} = 38.87.$$

So the object hits the water after 3.96 s with a speed of $38.9\,\mathrm{m\,s^{-1}}$. (These are, of course, the same as the answers found in Example 1.)

Solutions to the exercises in Section 4

1.

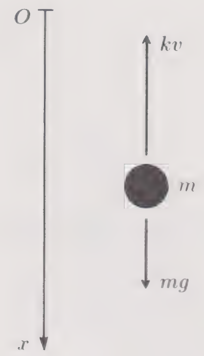


Figure 1

Choose an x-axis pointing downwards with the origin at the point where the parachute is opened. The forces acting on the parachutist are:

1. the force of gravity, of magnitude mg downwards;
2. air resistance, of magnitude kv upwards.

So Newton's second law gives

$$m\frac{dv}{dt} = mg - kv.$$

Using the separation of variables method to solve this differential equation, we obtain

$$\int \frac{m}{mg - kv}\,dv = \int 1\,dt + c$$

or $\quad -\dfrac{m}{k}\log_e(kv - mg) = t + c,$

since $v > mg/k$ and hence $kv - mg > 0$ throughout the motion. Using the initial condition $v = 2mg/k$ at $t = 0$, we have

$$c = -\frac{m}{k}\log_e(mg).$$

Therefore

$$-\frac{m}{k}\log_e(kv - mg) = t - \frac{m}{k}\log_e(mg)$$

or $\quad \log_e(kv - mg) = -\dfrac{kt}{m} + \log_e(mg).$

By taking exponentials, we obtain

$$kv - mg = mge^{-kt/m},$$

which can be rearranged to give

$$v = \frac{mg}{k}(1 + e^{-kt/m}).$$

Notice that in this case the velocity decreases from its initial value to the terminal speed $v_{\mathrm{T}} = mg/k$ (see Figure 2).

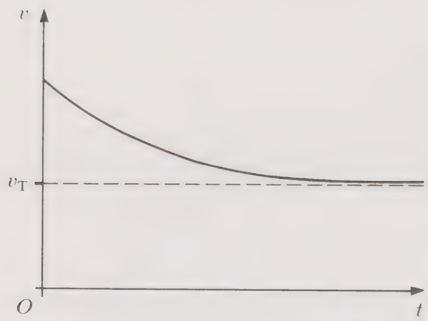


Figure 2

2.

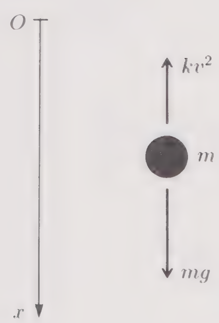


Figure 3

Choose an x-axis pointing vertically downwards with the origin at the point of projection. The forces acting on the particle are

1. the force of gravity, of magnitude mg downwards;
2. air resistance, of magnitude kv^2 upwards.

(i) Using Newton's first law, when the particle is moving with its terminal speed v_T the total force acting on the particle is zero. Hence

$$mg - kv_T{}^2 = 0,$$

that is,

$$v_T = \sqrt{\frac{mg}{k}}.$$

(ii) We wish to find v in terms of t, and so use $a = dv/dt$ in Newton's second law. This produces

$$m\frac{dv}{dt} = mg - kv^2,$$

or (writing $kv_T{}^2$ in place of mg, from part (i), and τ for v_T/g)

$$\tau v_T \frac{dv}{dt} = v_T{}^2 - v^2.$$

The method of separation of variables then gives

$$\int \frac{\tau v_T}{v_T{}^2 - v^2}\, dv = \int 1\, dt + c_1.$$

Using the first of the integrals given in the Hint yields

$$\frac{\tau}{2}\log_e\left(\frac{v_T + v}{v_T - v}\right) = t + c_1,$$

assuming that $v < v_T$ throughout the motion. The initial condition $v = 0$ at $t = 0$ gives $c_1 = 0$. Hence we have

$$\log_e\left(\frac{v_T + v}{v_T - v}\right) = \frac{2t}{\tau}.$$

Taking exponentials,

$$\frac{v_T + v}{v_T - v} = e^{2t/\tau}.$$

Rearranging this equation gives

$$v = v_T\left(\frac{e^{2t/\tau} - 1}{e^{2t/\tau} + 1}\right).$$

(iii) We wish to find v in terms of x, and so use $a = v\,dv/dx$ in Newton's second law. This produces

$$mv\frac{dv}{dx} = mg - kv^2$$

or (writing $kv_T{}^2$ in place of mg)

$$\frac{mv}{k}\frac{dv}{dx} = v_T{}^2 - v^2.$$

The method of separation of variables then gives

$$\frac{m}{k}\int \frac{v}{v_T{}^2 - v^2}\, dv = \int 1\, dx + c_2.$$

Using the second integral given in the Hint yields

$$-\frac{m}{2k}\log_e\left(v_T{}^2 - v^2\right) = x + c_2,$$

assuming that $v < v_T$ throughout the motion. The initial condition $v = 0$ when $x = 0$ leads to

$$c_2 = -\frac{m}{2k}\log_e\left(v_T{}^2\right).$$

Hence we have

$$-\frac{m}{2k}\log_e\left(v_T{}^2 - v^2\right) = x - \frac{m}{2k}\log_e\left(v_T{}^2\right)$$

or $\quad \log_e\left(v_T{}^2 - v^2\right) = -\dfrac{2kx}{m} + \log_e\left(v_T{}^2\right).$

Taking exponentials,

$$v_T{}^2 - v^2 = v_T{}^2 e^{-2kx/m}.$$

Rearranging this equation gives

$$v^2 = v_T{}^2\left(1 - e^{-2kx/m}\right)$$

or $\quad v = v_T\left(1 - e^{-2kx/m}\right)^{1/2}.$

3. The magnitude of the force of air resistance is $R = k|v|^2 = kv^2$, whether v is positive or negative. The direction of the force is always opposite to that of the motion. The only other force acting on the particle is gravity, of magnitude mg downwards.

(i)

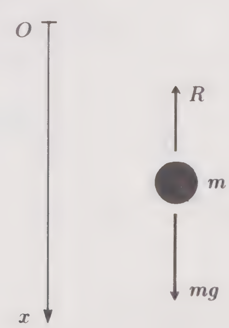


Figure 4

When the particle is moving *downwards*, air resistance acts upwards. The equation of motion is

$$m\frac{dv}{dt} = mg - R = mg - kv^2.$$

(ii)

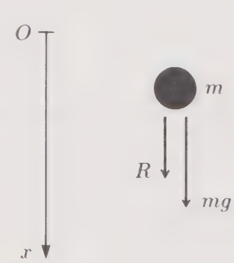


Figure 5

When the particle is moving *upwards*, air resistance acts downwards. The equation of motion is

$$m\frac{dv}{dt} = mg + R = mg + kv^2.$$

Solutions to the exercises in Section 5

1. The position function is

$$x(t) = t^4 - 3t^2 + 1,$$

so differentiation gives the velocity function

$$v(t) = \frac{dx}{dt} = 4t^3 - 6t$$

and the acceleration function

$$a(t) = \frac{dv}{dt} = 12t^2 - 6.$$

(i) At time $t = 1$,
the particle's position is $x(1) = -1$;
the particle's velocity is $v(1) = -2$;
the particle's speed is $|v(1)| = 2$;
the particle's acceleration is $a(1) = 6$.

(ii) The velocity is the rate of change of position which means that, as $v(1)$ is negative, the particle is moving in the direction of decreasing x at the instant $t = 1$. The acceleration is the rate of change of velocity which means that, as $a(1)$ is positive whereas $v(1)$ is negative, the particle is slowing down at the instant $t = 1$.

2. First consider the motion of particle A. Its acceleration is

$$a_A(t) = t + 2,$$

so its velocity is

$$v_A(t) = \int (t+2)\,dt + c_1 = \tfrac{1}{2}t^2 + 2t + c_1.$$

The initial condition $v_A(0) = 1$ leads to $c_1 = 1$, and so

$$v_A(t) = \tfrac{1}{2}t^2 + 2t + 1.$$

Hence

$$x_A(t) = \int (\tfrac{1}{2}t^2 + 2t + 1)\,dt + c_2 = \tfrac{1}{6}t^3 + t^2 + t + c_2.$$

Using the initial condition $x_A(0) = 0$, we obtain $c_2 = 0$. Therefore the position function of particle A is

$$x_A(t) = \tfrac{1}{6}t^3 + t^2 + t.$$

Now consider the motion of particle B. Its acceleration is

$$a_B(t) = t - 1.$$

Hence its velocity is

$$v_B(t) = \int (t-1)\,dt + c_3 = \tfrac{1}{2}t^2 - t + c_3.$$

The initial condition $v_B(0) = 0$ gives $c_3 = 0$. Hence

$$v_B(t) = \tfrac{1}{2}t^2 - t,$$

and so

$$x_B(t) = \int (\tfrac{1}{2}t^2 - t)\,dt + c_4 = \tfrac{1}{6}t^3 - \tfrac{1}{2}t^2 + c_4.$$

Using the initial condition $x_B(0) = 8$ leads to $c_4 = 8$. So the position function of particle B is

$$x_B(t) = \tfrac{1}{6}t^3 - \tfrac{1}{2}t^2 + 8.$$

The two particles collide when $x_A(t) = x_B(t)$, that is, when

$$\tfrac{1}{6}t^3 + t^2 + t = \tfrac{1}{6}t^3 - \tfrac{1}{2}t^2 + 8,$$

or $3t^2 + 2t - 16 = 0$.

This quadratic equation can be factorized as

$$(3t+8)(t-2) = 0,$$

so the particles collide at

$$t = -\tfrac{8}{3} \quad \text{or} \quad t = 2.$$

The particles are set moving at $t = 0$, so we can reject the negative time as being physically unrealistic. We therefore conclude that the particles collide at $t = 2$, when they are at position

$$x_B(2) = x_A(2) = \tfrac{1}{6} \times 2^3 + 2^2 + 2 = \tfrac{22}{3} = 7\tfrac{1}{3}.$$

3. The graphs of the acceleration and velocity functions for the car's motion are shown below.

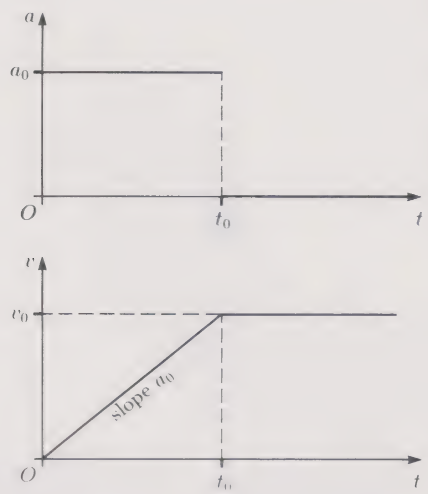


Figure 1

First we consider the period $0 \le t < t_0$ of constant non-zero acceleration. If this acceleration is a_0 then, either by integrating $a(t) = a_0$ or by using the formulas for constant acceleration, we have

$$v(t) = a_0 t \quad \text{and} \quad x(t) = \tfrac{1}{2}a_0 t^2.$$

Now $v(t_0) = a_0 t_0 = v_0$, and so

$$a_0 = \frac{v_0}{t_0}.$$

Hence, the velocity and position during the first phase of the motion are

$$v(t) = \frac{v_0 t}{t_0} \qquad (0 \le t < t_0)$$

and

$$x(t) = \frac{v_0 t^2}{2t_0} \qquad (0 \le t < t_0).$$

In particular, its position at the end of this period is

$$x_0 = x(t_0) = \tfrac{1}{2}v_0 t_0.$$

Now consider the period $t \ge t_0$ of constant velocity v_0. By first integrating $v(t) = v_0$ and then using the initial condition $x(t_0) = \frac{1}{2}v_0 t_0$, we obtain

$$x(t) = \tfrac{1}{2}v_0 t_0 + v_0(t - t_0) = v_0 t - \tfrac{1}{2}v_0 t_0 \qquad (t \ge t_0).$$

Summarizing, for the whole motion we have

$$x(t) = \begin{cases} v_0 t^2/(2t_0) & (0 \le t < t_0), \\ v_0 t - \frac{1}{2}v_0 t_0 & (t \ge t_0), \end{cases}$$

$$v(t) = \begin{cases} v_0 t/t_0 & (0 \le t < t_0), \\ v_0 & (t \ge t_0). \end{cases}$$

The graph of the position function is shown below.

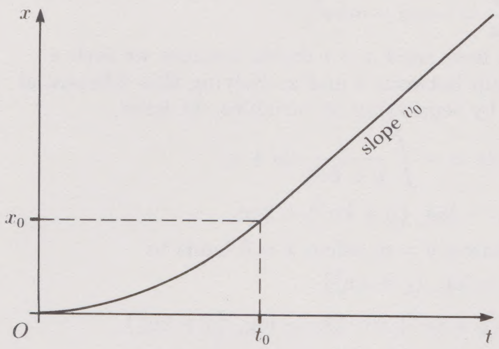


Figure 2

4. The acceleration a is constant, and so from the definition of acceleration as rate of change of velocity, we have

$$a = \frac{\text{increase in velocity}}{\text{time interval}} = \tfrac{3}{2}\,\mathrm{m\,s^{-2}}.$$

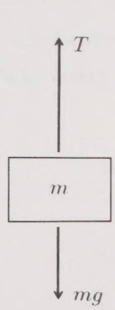


Figure 3

Denoting the mass of the lift by m and the tension in the cable by T, Newton's second law relative to an upward-pointing x-axis gives

$$T - mg = ma,$$

so that

$$\begin{aligned} T &= ma + mg \\ &= 1000(1.5 + 9.81) \\ &= 11310\,\mathrm{N}. \end{aligned}$$

5. By Newton's second law the force on the particle has x-component

$$F = ma = a,$$

since $m = 1$. Now

$$x = e^{-3t}\sin 4t,$$

so $v = \dfrac{dx}{dt} = 4e^{-3t}\cos 4t - 3e^{-3t}\sin 4t.$

Hence

$$\begin{aligned} a = \frac{dv}{dt} &= 4(-4e^{-3t}\sin 4t - 3e^{-3t}\cos 4t) \\ &\quad - 3(4e^{-3t}\cos 4t - 3e^{-3t}\sin 4t) \\ &= -7e^{-3t}\sin 4t - 24e^{-3t}\cos 4t. \end{aligned}$$

Now

$$\begin{aligned} -25x - 6v = &-25e^{-3t}\sin 4t \\ &- 6(4e^{-3t}\cos 4t - 3e^{-3t}\sin 4t) \\ = &-7e^{-3t}\sin 4t - 24e^{-3t}\cos 4t. \end{aligned}$$

Hence

$$\begin{aligned} F = a &= -7e^{-3t}\sin 4t - 24e^{-3t}\cos 4t \\ &= -25x - 6v. \end{aligned}$$

6.

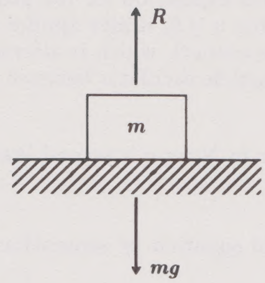


Figure 4

(i) Relative to an upward-pointing x-axis the total force has x-component

$$F = R - mg.$$

When the plane has an upward acceleration of magnitude a, Newton's second law gives

$$R - mg = ma,$$

which leads to

$$R = m(g + a).$$

Notice that the magnitude R of the reaction force is greater than the object's weight mg.

(ii) Relative to a downward-pointing x-axis the total force has x-component

$$F = mg - R.$$

When the plane has a downward acceleration of magnitude a, Newton's second law gives

$$mg - R = ma,$$

which leads to

$$R = m(g - a).$$

The condition $a < g$ ensures that this magnitude is positive. Notice that the reaction R is less than the object's weight mg. In the limiting case $a = g$ we have $R = 0$, so that there is then no reaction on the object due to the plane. If the downward acceleration magnitude a of the plane were greater than g then the object would leave the surface of the plane and fall freely under gravity.

7. As the force depends on x, we use $a = v\,dv/dx$ in Newton's second law, giving

$$v\frac{dv}{dx} = -\omega^2 x.$$

Separating variables and integrating this differential equation gives

$$\int v\,dv = c_1 - \omega^2 \int x\,dx$$

or $\tfrac{1}{2}v^2 = c_1 - \tfrac{1}{2}\omega^2 x^2.$

The initial condition $v = v_0$ at $x = 0$ leads to $c_1 = \tfrac{1}{2}v_0^2$, so that

$$v = \pm\sqrt{v_0^2 - \omega^2 x^2}.$$

We look first at the case when v is non-negative, since when $x = 0$ we have $v = v_0 > 0$. Then x can be found in terms of t by using $v = dx/dt$. We have

$$\frac{dx}{dt} = \sqrt{v_0^2 - \omega^2 x^2},$$

so $\displaystyle\int 1\,dt = \int \frac{1}{\sqrt{v_0^2 - \omega^2 x^2}}\,dx + c_2,$

or $t = \dfrac{1}{\omega}\arcsin\left(\dfrac{wx}{v_0}\right) + c_2.$

The initial condition $x = 0$ at $t = 0$ gives $c_2 = 0$. Hence

$$t = \frac{1}{\omega}\arcsin\left(\frac{wx}{v_0}\right)$$

or $x = \dfrac{v_0}{\omega}\sin(\omega t).$

Note that, although this expression for the position function was derived by assuming $v \geq 0$, it also applies when $v < 0$. In fact, $v = dx/dt = v_0 \cos(\omega t)$, which is alternately positive and negative as the particle oscillates between $x = v_0/\omega$ and $x = -v_0/\omega$.

8. Using $a = v\,dv/dx$ in Newton's second law gives

$$mv\frac{dv}{dx} = kxv^2.$$

Solving this differential equation by separation of variables gives

$$m\int \frac{1}{v}\,dv = k\int x\,dx + c,$$

or $\quad m\log_e v = \frac{1}{2}kx^2 + c,$

where we have assumed $v > 0$ because $v = v_0 > 0$ at $x = 0$. The initial condition $v = v_0$ when $x = 0$ gives $c = m\log_e v_0$. Hence

$$\log_e v = \frac{kx^2}{2m} + \log_e v_0.$$

Taking exponentials of both sides of this equation, we obtain

$$v = v_0 \exp\left(\frac{kx^2}{2m}\right).$$

9. Choose an x-axis pointing vertically upwards with the origin at the point of projection. We assume that the only force acting on the particles is gravity, so that the acceleration of both particles is $a = -g$. By integrating this equation and using the initial conditions, or by using the formulas for constant acceleration, the position of the first particle is

$$x_1 = v_0 t - \tfrac{1}{2}gt^2 \qquad (t \geq 0),$$

whereas the position of the second particle is

$$x_2 = v_0(t-T) - \tfrac{1}{2}g(t-T)^2 \qquad (t \geq 0).$$

The two particles will collide when $x_1 = x_2$, that is, when

$$v_0 t - \tfrac{1}{2}gt^2 = v_0(t-T) - \tfrac{1}{2}g(t-T)^2$$

or $\quad 0 = -v_0 T + gtT - \frac{1}{2}gT^2.$

Therefore

$$gt = \tfrac{1}{2}gT + v_0$$

or $\quad t = \dfrac{T}{2} + \dfrac{v_0}{g}.$

(Note: the condition $T < 2v_0/g$ given in the question ensures that the first particle is still in the air when the second one is thrown.)

10. We choose an x-axis pointing upwards with origin at the point of projection, so that x is the height of the particle above its starting point.

(i) During the *upward* motion, the forces on the particle are:

1. gravity, of magnitude mg downwards;
2. air resistance, of magnitude mkv^2 downwards.

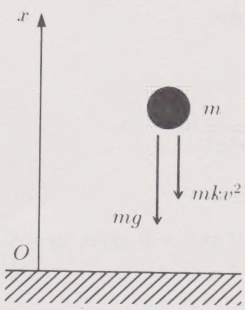


Figure 5

Newton's second law gives

$$mv\frac{dv}{dx} = -mg - mkv^2,$$

where we have used $a = v\,dv/dx$ because we seek a relationship between v and x. Solving this differential equation by separation of variables, we have

$$\int 1\,dx = -\int \frac{v}{g + kv^2}\,dv + c_1$$

or $\quad 2kx = -\log_e\left(g + kv^2\right) + 2kc_1.$

The condition $v = v_0$ when $x = 0$ leads to

$$2kc_1 = \log_e\left(g + kv_0^2\right).$$

So $\quad \log_e\left(g + kv^2\right) = -2kx + \log_e\left(g + kv_0^2\right).$

Taking exponentials of both sides produces

$$g + kv^2 = e^{-2kx}\left(g + kv_0^2\right),$$

which can be rearranged to give the required result

$$v^2 = \left(\frac{g}{k} + v_0^2\right)e^{-2kx} - \frac{g}{k}.$$

(ii) During the *downward* motion, the forces on the particle are:

1. gravity, of magnitude mg downwards;
2. air resistance, of magnitude mkv^2 upwards.

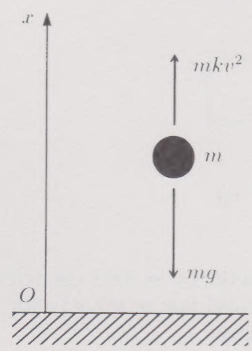


Figure 6

Newton's second law gives

$$mv\frac{dv}{dx} = -mg + mkv^2.$$

Solving this differential equation by separation of variables, we obtain

$$\int 1\,dx = -\int \frac{v}{g - kv^2}\,dv + c_2$$

or $\quad 2kx = \log_e(g - kv^2) + 2kc_2.$

We have assumed that $g - kv^2 > 0$ throughout the downward motion, or equivalently, that the particle's speed remains less than its terminal speed. The condition $v = -v_1$ when $x = 0$ (note that v_1 is the speed and v is negative as the particle is moving downwards, so we have $v = -v_1$ rather than $v = v_1$) gives

$$2kc_2 = -\log_e(g - kv_1^2).$$

So $\quad \log_e(g - kv^2) = 2kx + \log_e(g - kv_1^2).$

Taking exponentials of both sides produces

$$g - kv^2 = e^{2kx}(g - kv_1^2),$$

which can be rearranged to give the required result

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} - v_1^2\right)e^{2kx}.$$

(iii) At the highest point $x = x_{\max}$ and $v = 0$. For the upward motion this gives

$$\left(\frac{g}{k} + v_0^2\right) e^{-2kx_{\max}} = \frac{g}{k}, \qquad (1)$$

whereas for the downward motion this condition gives

$$\left(\frac{g}{k} - v_1^2\right) e^{2kx_{\max}} = \frac{g}{k}. \qquad (2)$$

We need to eliminate $x_{\max}$ between these two equations. From Equation (1),

$$e^{-2kx_{\max}} = \frac{g}{g + kv_0^2},$$

or $$e^{2kx_{\max}} = \frac{g + kv_0^2}{g}.$$

Substituting this into Equation (2),

$$(g - kv_1^2)(g + kv_0^2) = g^2.$$

We wish to make $1/v_1^2$ the subject of this equation. Working towards this end, we have

$$g - kv_1^2 = \frac{g^2}{g + kv_0^2}.$$

Therefore

$$kv_1^2 = g - \frac{g^2}{g + kv_0^2} = \frac{kgv_0^2}{g + kv_0^2},$$

so $$\frac{1}{v_1^2} = \frac{g + kv_0^2}{gv_0^2} = \frac{1}{v_0^2} + \frac{k}{g},$$

which is the required result.

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